Chapter 3 Circuit Analysis Methods

3.1 Two Types of Constraints and Circuit Equations

3.1.1 Two Types of Constraints

A circuit is formed by connecting components in a specific manner, and in any lumped-element circuit, the currents and voltages must satisfy two types of constraints related to the properties of the components and the way the circuit is connected.

(1) Constraints related to the properties of the components: The voltage-current relationships of the components provide linear constraints on the voltages and currents in each branch (such as Ohm's Law $U=RI$). These constraints are independent of the circuit's connectivity.

(2) Constraints related to the circuit connection methods: Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL) respectively provide linear constraints on the branch currents and branch voltages with a specific connection method. These constraints are independent of the properties of the components.

Kirchhoff's Current Law: At any node in a circuit, the sum of currents flowing into the node at any given moment is equal to the sum of currents flowing out of the node.

Kirchhoff's Voltage Law: In any closed loop, the algebraic sum of the voltage drops across all components is equal to the algebraic sum of the electromotive forces. In other words, when traversing a closed loop and returning to the starting point, the algebraic sum of the voltage drops across each segment is always equal to zero.

In any lumped-element circuit, the voltages and currents must simultaneously satisfy these two types of constraints. Therefore, the fundamental method for circuit analysis involves, based on the circuit's structure and parameters, formulating Kirchhoff's Current Law (KCL), Kirchhoff's Voltage Law (KVL), and component Voltage-Current Relationship (VCR) equations that reflect the two types of constraint relationships. These equations, collectively known as circuit equations, are then solved to obtain the solutions for the voltages and currents in the circuit.

3.1.2 Circuit Equations

For a circuit with b branches and n nodes, the circuit equations have the following characteristics:

(1) The VCR equations for the b branches are independent of each other.

(2) The KCL equations for any $n-1$ nodes are independent of each other, and the number of independent nodes is $n-1$.

(3) The KVL equations for any $b-n+1$ loops are independent of each other, and

the number of independent loops is equal to the number of meshes, which is $b-n+1$.

(4) The total number of independent circuit equations is $b+(n-1)+(b-n+1)=2b$. These equations represent the most fundamental circuit equations and serve as the basic foundation for circuit analysis.

(5) The VCR equations for independent power sources directly provide the voltage or current for the respective branch, thereby reducing constraint equations and variables in the solution process.

Example 3.1 In the circuit shown in Figure 3.1.1, $u_S=0.05\cos t$ (V), calculate the voltages and currents for each branch.

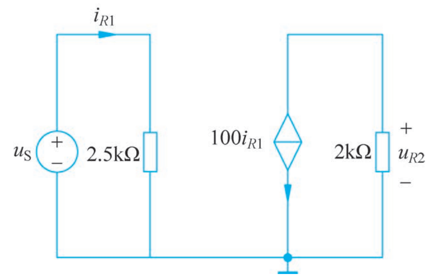


Figure 3.1.1 Example 3.1 circuit

Solution: The circuit has 4 branches, 2 independent nodes, and 2 meshes, which can list 4 VCR equations, 2 KCL equations and 2 KVL equations.

4 VCR equations:

$$u_S=0.05\cos t$$
$$u_{R1}=2.5i_{R1}$$
$$i_{CCCS}=100i_{R1}$$
$$u_{R2}=2i_{R2}$$

2 KCL equations:

$$i_U+i_{R1}=0$$
$$i_{CCCS}+i_{R2}=0$$

2 KVL equations:

$$u_{R1}-u_S=0$$
$$u_{R2}-u_{CCCS}=0$$

By solving the above equations, we can obtain:

$$u_{R1}=0.05\cos t\,(\text{V})$$
$$i_{R1}=\frac{0.05\cos t}{2.5}=0.02\cos t\,(\text{mA})$$
$$i_U=-0.02\cos t\,(\text{mA})$$
$$i_{CCCS}=100\times 0.02\cos t=2\cos t\,(\text{mA})$$
$$i_{R2}=-2\cos t\,(\text{mA})$$
$$u_{CCCS}=u_{R2}=2\times(-2\cos t)=-4\cos t\,(\text{V})$$

Example 3.2 In the circuit shown in Figure 3.1.2, the switch is closed at $t=0$. Given that the initial voltage across the capacitor is $u_C(0)=1$V, calculate the voltages and currents in each branch for $t\geqslant 0$.

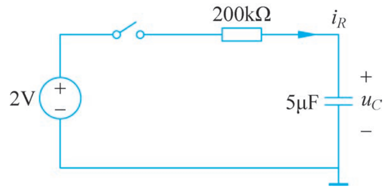


Figure 3. 1. 2 Example 3. 2 circuit

Solution: After the switch is closed, it consists of 3 branches, 2 independent nodes, and 1 mesh. We can formulate 3 VCR equations, 2 KCL equations, and 1 KVL equation.

3 VCR equations:

$$U_S = 2\text{V}$$

$$u_R = 200 \times 10^3 i_R$$

$$i_C = 5 \times 10^{-6} \frac{\mathrm{d}u_C}{\mathrm{d}t}$$

2 KCL equations:

$$i_U + i_R = 0$$

$$-i_R + i_C = 0$$

1 KVL equation:

$$u_R + u_C - U_S = 0$$

Loop KVL equation:

$$200 \times 10^3 \times 5 \times 10^{-6} \frac{\mathrm{d}u_C}{\mathrm{d}t} + u_C = \frac{\mathrm{d}u_C}{\mathrm{d}t} + u_C = 2$$

$$u_C(0) = 1$$

The solution to a first-order nonhomogeneous linear differential equation consists of two parts: $u_C = u_{Ch} + u_{Cp}$. The general solution u_{Ch} to the corresponding homogeneous linear differential equation is given by:

$$u_{Ch} = K\mathrm{e}^{st}, \quad t \geqslant 0 \tag{3.1.1}$$

In the equation: s represents the characteristic roots of the characteristic equation; K is an undetermined constant. For the characteristic equation $s + 1 = 0$, the characteristic root is $s = -1$.

A particular solution u_{Cp} for the first-order nonhomogeneous linear differential equation generally takes the same form as the excitation. Assuming $u_{Cp} = C$, substituting it into the first-order nonhomogeneous linear differential equation $u_{Cp} = 2$. Hence, we obtain:

$$u_C = u_{Ch} + u_{Cp} = K\mathrm{e}^{-t} + 2, \quad t \geqslant 0 \tag{3.1.2}$$

Solving for the undetermined constant K with the initial voltage $u_C(0) = 1$, setting $t = 0$, we have:

$$u_C(0) = K + 2 = 1 \rightarrow K = -1 \rightarrow u_C = 2 - \mathrm{e}^{-t}(\text{V}), \quad t \geqslant 0 \tag{3.1.3}$$

After solving for the voltage across the capacitor, we can determine the voltage across the resistor:

$$u_R = U_S - u_C = \mathrm{e}^{-t}(\text{V}), \quad t \geqslant 0 \tag{3.1.4}$$

The currents flowing through each component are:

$$i_C = 5\times10^{-6}\frac{du_C}{dt} = 5\times10^{-6}e^{-t}(\text{A}) = 5e^{-t}(\mu\text{A}),\quad t\geqslant 0 \tag{3.1.5}$$

$$i_R = i_C = 5e^{-t}(\mu\text{A}),\quad t\geqslant 0 \tag{3.1.6}$$

$$i_U = -i_R = -5e^{-t}(\mu\text{A}),\quad t\geqslant 0 \tag{3.1.7}$$

3.2 The Three-Element Method for First-Order Circuits

3.2.1 First-Order RC Circuit

In Figure 3.2.1(a), we have a simple voltage-source resistor-capacitor (RC) circuit. Let's write the Kirchhoff's Current Law (KCL) equation for the top node.

$$i(t) = \frac{u_C}{R} + C\frac{du_C}{dt} \tag{3.2.1}$$

Equation (3.2.1) can be rewritten as:

$$\frac{du_C}{dt} + \frac{u_C}{RC} = \frac{i(t)}{C} \tag{3.2.2}$$

To find $u_C(t)$, it is necessary to solve a nonhomogeneous linear first-order ordinary differential equation. We use the method of finding homogeneous and particular solutions to solve this equation. Let $u_{Ch}(t)$ be an arbitrary solution of the homogeneous Equation (3.2.3) associated with the inhomogeneous Equation (3.2.2).

$$\frac{du_C}{dt} + \frac{u_C}{RC} = 0 \tag{3.2.3}$$

Setting the original driving function (here denoted as $i(t)$) in the non-homogeneous equation to zero yields the corresponding homogeneous equation. Subsequently, let $u_{Cp}(t)$ be an arbitrary solution to Equation (3.2.2). Finally, add the two solutions together to obtain the complete solution:

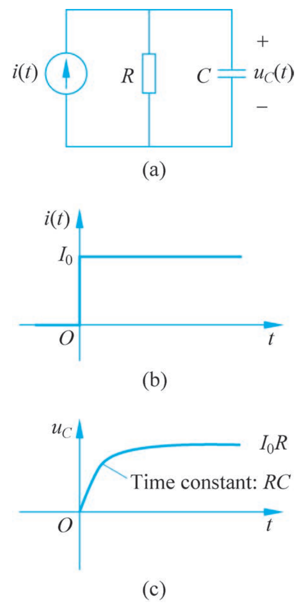


Figure 3.2.1 The transient response of capacitor charging

$$u_C(t) = u_{Ch}(t) + u_{Cp}(t) \tag{3.2.4}$$

Equation (3.2.4) represents the general or complete solution of Equation (3.2.2). Here, $u_{Ch}(t)$ is referred to as the homogeneous solution, while $u_{Cp}(t)$ is called the particular solution. In the context of circuit response, the homogeneous solution can also be termed as the circuit's natural response since it solely depends on the internal energy storage properties of the circuit and is independent of external inputs. The particular solution can also be referred to as the forced response or forced solution, as it is determined by the external inputs to the circuit.

To make the problem more specific, let's assume that the current source is a step function.

$$i(t)=I_0,\quad t>0 \tag{3.2.5}$$

As shown in Figure 3.2.1(b), it is further assumed that before the step current is applied, the voltage across the capacitor is 0. From a mathematical perspective, this constitutes the initial condition.

$$u_C=0,\quad t<0 \tag{3.2.6}$$

The method to solve for the homogeneous solution and particular solution typically involves three steps.

The first step involves finding the homogeneous solution, denoted as $u_{Ch}(t)$, which is given by:

$$\frac{du_{Ch}}{dt}+\frac{u_{Ch}}{RC}=0 \tag{3.2.7}$$

Assuming the form of the solution is:

$$u_{Ch}=Ae^{st} \tag{3.2.8}$$

To substitute Equation (3.2.8) into Equation (3.2.7), we obtain:

$$Ase^{st}+\frac{Ae^{st}}{RC}=0 \tag{3.2.9}$$

The equation cannot determine the value of A. However, by excluding the special case of $A=0$, we obtain:

$$se^{st}+\frac{e^{st}}{RC}=0 \tag{3.2.10}$$

For finite values of s and t, the term e^{st} will never be equal to zero. Therefore, this factor can be eliminated, resulting in:

$$s=-\frac{1}{RC} \tag{3.2.11}$$

Equation (3.2.10) is the characteristic equation of the system, and $s=-1/(RC)$ is a root of this characteristic equation. Now, knowing that the homogeneous solution has the following form:

$$u_{Ch}=Ae^{-t/RC} \tag{3.2.12}$$

The product RC has the dimension of time and is referred to as the time constant of the circuit.

The second step involves finding a particular solution, denoted as u_{Cp}, which satisfies the original differential equation. It does not have to satisfy the initial conditions, i.e., it is required to satisfy the equation:

$$I_0=\frac{u_{Cp}}{R}+C\frac{du_{Cp}}{dt} \tag{3.2.13}$$

Since I_0 is a constant for $t>0$, an acceptable particular solution is also a constant, namely:

$$u_{Cp}=K \tag{3.2.14}$$

To prove this, substitute it into Equation (3.2.13), yielding:

$$I_0=\frac{K}{R}+0 \tag{3.2.15}$$

$$K=I_0R \tag{3.2.16}$$

Since Equation (3.2.15) can be used to determine K, there is confidence in the assumption about the particular solution form, that is, Equation (3.2.13) is correct. Therefore, the particular solution is:

$$u_{Cp}=I_0R \tag{3.2.17}$$

The third step involves determining the complete solution. The complete solution is the sum of the homogeneous solution and the particular solution.

$$u_C=A\mathrm{e}^{-t/RC}+I_0R \tag{3.2.18}$$

The only remaining unknown constant is A, and it can be determined using the initial conditions. Equation (3.2.6) applies to $t<0$, while Equation (3.2.18) applies to $t>0$. As the instantaneous jump in capacitor voltage requires an infinite pulse current, the capacitor voltage must be continuous for finite current. The circuit cannot provide an infinitely large current, thus it is reasonable to assume that the voltage across the capacitor u_C is continuous. Consequently, the solutions for the positive and negative time intervals are equal at the moment $t=0$.

$$0=A+I_0R \tag{3.2.19}$$

Therefore, we have:

$$A=-I_0R \tag{3.2.20}$$

The complete solution for $t>0$ is:

$$u_C=-I_0R\mathrm{e}^{-t/RC}+I_0R \tag{3.2.21}$$

Or:

$$u_C=I_0R(1-\mathrm{e}^{-t/RC}) \tag{3.2.22}$$

Plot the graph of the complete solution, as shown in Figure 3.2.1(c).

Here are some explanatory notes to deepen the understanding:

① Note that the capacitor voltage starts from 0 at $t=0$, and after a considerable time t, reaches its final value of I_0R. The growth process from 0 to I_0R has a time constant of RC. The final value of the capacitor voltage, I_0R, indicates that all the current emitted by the current source flows through the resistor. The capacitor appears as an open circuit.

② The initial value of the capacitor voltage being 0 implies that at $t=0$, all the current emitted by the current source must flow through the capacitor, and there is no current through the resistor. Therefore, at $t=0$, the capacitor appears as an instantaneous short circuit.

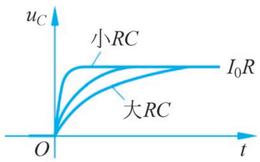


Figure 3.2.2 Time constant

③ Now the physical significance of the time constant RC becomes apparent. As shown in Figure 3.2.2, it is a factor that characterizes transient behavior, determining the speed at which the transition process concludes.

The capacitor is now fully charged, assuming the current source is suddenly set to zero, as shown in Figure 3.2.3(a). For convenience, the time axis is redefined in the figure so that the current source is turned off at $t=0$. The circuit used to analyze the transient process of RC discharge now only contains a resistor and a capacitor, as shown in Figure 3.2.3(c). The initial condition describing the voltage across the capacitor at the beginning of the experiment is given by:

$$u_C = I_0 R, \quad t < 0 \tag{3.2.23}$$

In this scenario, the RC discharge process is the same as that of a circuit containing only a resistor and a capacitor, with the initial capacitor voltage $u_{C(0)} = I_0 R$.

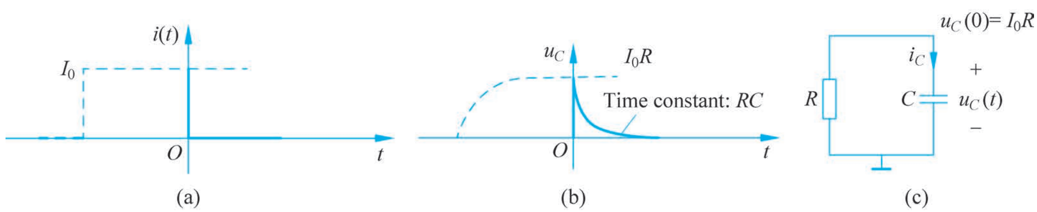


Figure 3.2.3 Transient process of capacitor discharge

Since the drive current is 0, the differential equation for $t>0$ is

$$0 = \frac{u_C}{R} + \frac{C\,du_C}{dt}$$

As before, the homogeneous solution is

$$u_{Ch} = A e^{-t/RC} \tag{3.2.24}$$

However, the particular solution is now 0 because there is no forced input, so Equation (3.2.24) is the complete solution. In other words

$$u_C = u_{Ch} = A e^{-t/RC} \tag{3.2.25}$$

Let Equation (3.2.23) and Equation (3.2.24) be equal at time $t=0$, obtain

$$I_0 R = A \tag{3.2.26}$$

Therefore, when $t>0$, the waveform of the capacitor voltage is

$$u_C = I_0 R e^{-t/RC} \tag{3.2.27}$$

The solution diagram is shown in Figure 3.2.3(b).

Generally speaking, a circuit composed of a resistor and a capacitor. If the initial value of the capacitor voltage is $u_C(0)$, the waveform of the capacitor voltage at $t>0$ is

$$u_C = u_C(0)e^{-t/RC} \tag{3.2.28}$$

3.2.2 Properties of Exponent

Since decay exponents often occur in the solution of simple RC and RL transient problems, discussing certain properties of these functions here will be beneficial for plotting their graphs.

The general form of an exponential function is

$$x = Ae^{-t/\tau} \tag{3.2.29}$$

The starting slope of the exponent is

$$\left.\frac{dx}{dt}\right|_{t=0} = \frac{-A}{\tau} \tag{3.2.30}$$

Therefore, by extending the initial slope of the curve as a straight line to intersect with the time axis, the intersection occurs at $t=\tau$, which is independent of the value of A, as shown in Figure 3.2.4(a).

In addition, notice that when $t=\tau$, the function in Equation (3.2.29) becomes

$$x(t=\tau) = \frac{A}{e} \tag{3.2.31}$$

In other words, the function reaches its initial value of $1/e$, regardless of the value of A. This is depicted on the exponential curve in Figure 3.2.4(b).

Since $e^{-5}=0.0067$, it is generally assumed that t is greater than 5 time constants, i.e.

$$t > 5\tau \tag{3.2.32}$$

The function is essentially already 0, which implies that the transient process is assumed to have concluded.

We shall see later that these properties of the time constant τ are very useful for estimating roughly the duration of exponential growth or decay.

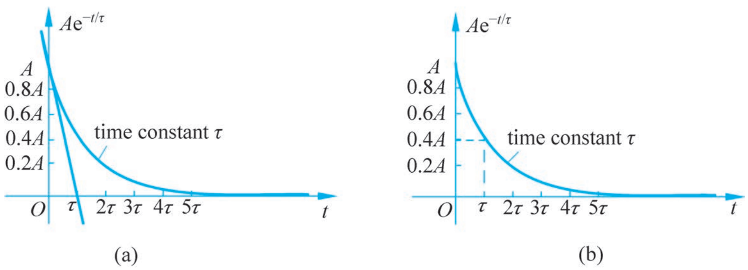


Figure 3.2.4 Properties of exponents

3.3 Superposition Theorem and Its Application

3.3.1 Superposition Theorem

The voltage or current of any branch generated by the combined action of m

independent voltage sources and n independent current sources in a circuit is equal to the algebraic sum of the corresponding branch voltage or current components generated by the separate action of each independent power source, where all branch voltages or current components take the same reference direction.

$$y=\sum_{i=1}^{m+n} y_i=\sum_{i=1}^{m+n} K_i x_i \tag{3.3.1}$$

$$y_i = y \mid \cap_{j\neq i} x_j = 0 = K_i x_i, \quad i,j=1,2,\cdots,m+n \tag{3.3.2}$$

$$y = u \text{ or } i$$

$$x_i = u_{Si} \text{ or } i_{Si} \tag{3.3.3}$$

When a particular independent source acts alone, it is equivalent to setting all other independent sources in the circuit to zero, i. e., short-circuiting independent voltage sources and open-circuiting independent current sources. Controlled sources, however, do not fall into the category of acting alone or being set to zero.

3.3.2 Application of Superposition Theorem

To determine the voltage or current in any branch of a circuit due to the combined action of several independent sources, it is only needed to calculate the respective branch voltage or current components produced by each independent source acting alone and then superimpose these components.

Example 3.3 In the circuit shown in Figure 3.3.1, $u_S = 0.01\cos t$ (V), determine the voltage u_O.

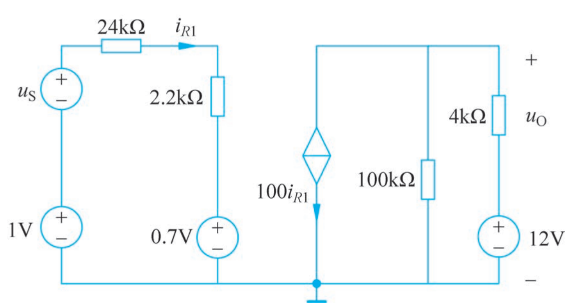


Figure 3.3.1 Example 3.3 circuit

Solution: When all DC power supplies act (as shown in Figure 3.3.2), there is

$$I_{R1}=\frac{1-0.7}{24+2.2}\approx 0.0115(\text{mA})$$

$$U_O=\frac{100}{4+100}\times 12-100\times 0.0115\left(\frac{4\times 100}{4+100}\right)=11.5-11.5\times 3.85\approx 7.08(\text{V})$$

When AC power supply acts alone (as shown in Figure 3.3.3):

$$i_{R1}=\frac{0.01\cos t}{24+2.2}\approx 0.0004\cos t(\text{mA})$$

$$u_O = -100 \times 0.0004\cos t\left(\frac{4 \times 100}{4 + 100}\right) = -0.04\cos t \times 3.85 \approx -0.15\cos t\,(V)$$

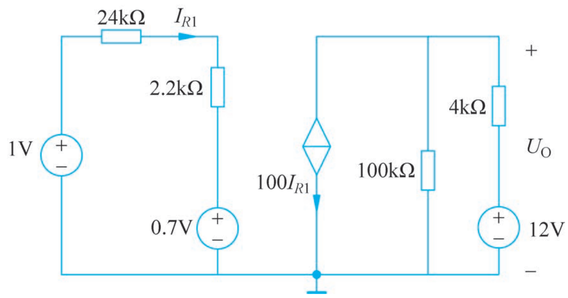


Figure 3.3.2 Example 3.3(a) Diagram

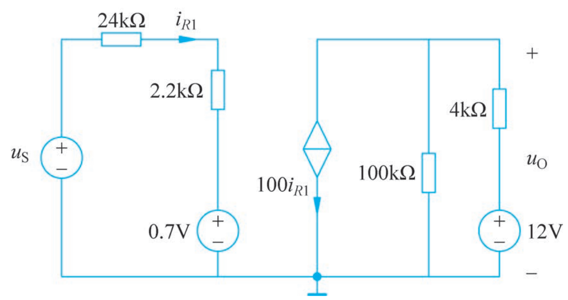


Figure 3.3.3 Example 3.3(b) diagram

The output voltage after superposition is

$$u_O = U_O + u_O = 7.08 - 0.15\cos t\,(V)$$

3.4 Network Equivalence with the Application of Thevenin's Theorem and Norton's Theorem

3.4.1 Network Equivalence

A single-port network is a circuit that has only one external port. A network that is connected to other circuits only through two terminals is referred to as a two-terminal network. When the port characteristics of the two-terminal network are emphasized without concern for the internal situation of the network, the two-terminal network is called a single-port network, or single port for short. Active single-port contains an independent power supply, generally represented by N. Passive single port does not contain an independent power supply and is generally represented by N_0.

The external circuit refers to the other parts of the circuit connected by a single port, and the external characteristics of the single port are determined by the port VCR.

If two single port ports have the same VCR, a single port network can be referred to as external equivalence. Two equivalent single ports have the same effect on the external circuit, but their internal structural parameters can be completely different. The single port circuit is shown in Figure 3.4.1.

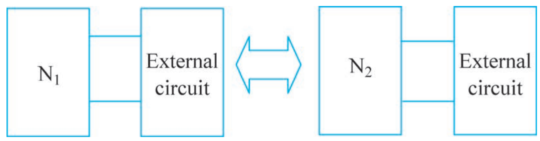


Figure 3.4.1 Single-port circuit

A single-port equivalent circuit is the simplest circuit that can reflect a port VCR.

Example 3.4 Are the two single-port networks shown in Figure 3.4.2 equivalent?

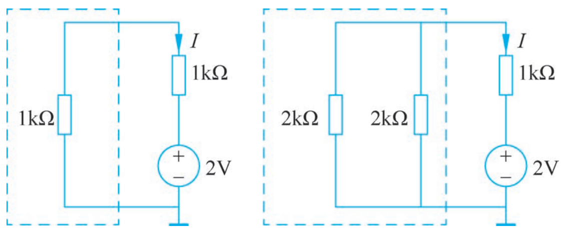


Figure 3.4.2 Example 3.4 circuit

Solution: For both single-port networks, the terminal (VCR) is equivalent, characterized by $U=I$.

(1) For both single-port networks, the effect on the external circuit is $I=-1$mA;

(2) The internal one of the two single ports is a 1kΩ resistor, and the other is two 2kΩ resistors in parallel.

Example 3.5 Are the two single-port networks as shown in Figure 3.4.3 equivalent?

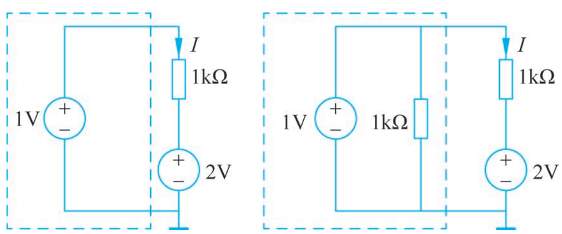


Figure 3.4.3 Example 3.5 circuit

Solution: For both single-port networks, the terminal (VCR) is equivalent, characterized by $U=1$V.

(1) For both single-port networks, the effect on the external circuit is $I=-1$mA;

(2) The internal one of the two single ports is a 1V voltage source, and the other is a 1V voltage source in parallel with a 1kΩ resistor. The single-port equivalent circuit is

the 1V voltage source.

3.4.2 Thevenin's Theorem and Norton's Theorem

1. Thevenin's theorem

The port characteristic of any active resistance single port N is equivalent to a series of voltage source resistors, and this circuit is called Thevenin equivalent Road, as shown in Figure 3.4.4. Where the voltage source u_{oc} is the port open voltage of N. Resistance R_o is the equivalent resistance of single port N_0 of passive resistance which corresponds to the original network N with all independent sources set to zero. This resistance is known as the Thevenin equivalent resistance.

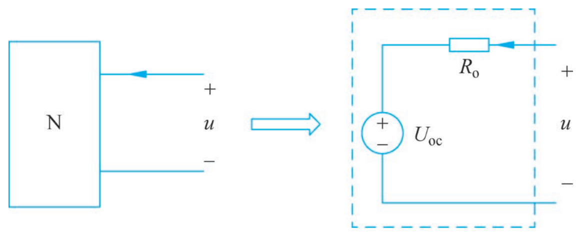


Figure 3.4.4 Thevenin equivalent circuit

The port VCR of the Thevenin equivalent circuit is

$$u = u_{oc} + R_o i \tag{3.4.1}$$

The meaning of Thevenin's theorem:

(1) It is established that any active resistor single-port can be equivalently represented as a series connection of a voltage source and a resistor;

(2) Provides methods for simplifying an active resistance single port.

2. Norton's theorem

The port characteristic of any active resistance single port N is equivalent to the current source resistance in parallel, which is called Norton equivalent circuit, as shown in Figure 3.4.5. Where, the current source i_{sc} is the port short circuit current of N, and the resistance R_o is the equivalent resistance of the passive single-port network N_0, obtained by setting all independent sources within network N to zero. This resistance is referred to as the Norton equivalent resistance, and it is also known as the Thevenin equivalent resistance.

The port VCR of the Norton equivalent circuit is

$$i = -i_{sc} + u/R_o \tag{3.4.2}$$

The meaning of Norton's theorem:

(1) The single port of any active resistance is equivalent to the parallel current source resistance;

(2) Provides the method of simplifying the active resistance single port.

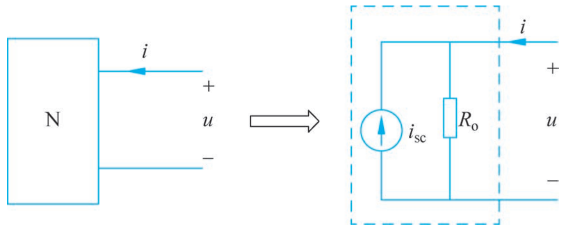


Figure 3.4.5 Norton equivalent circuit

3. The equivalence of Thevenin's theorem and Norton's theorem

1) The equivalent transformation of the Thevenin/ Norton equivalent circuit

The port VCR is the same if an active resistance single port N can be equivalent to both the Thevenin and Norton equivalent circuits.

Port VCR of Thevenin equivalent circuit:

$$u = u_{oc} + R_o i$$

Port VCR of Norton equivalent circuit:

$$i = -i_{sc} + u/R_o$$

$$u = u_{oc} + R_o i \xrightarrow{R_o \neq 0} i = -\frac{u_{oc}}{R_o} + \frac{1}{R_o}u = -i_{sc} + \frac{1}{R_o}u \qquad (3.4.3)$$

In the formula, $i_{sc} = \frac{u_{oc}}{R_o}$.

As long as the resistance of the Thevenin equivalent circuit is not zero (not a voltage source branch), it can be converted to a Norton equivalent circuit.

$$i = -i_{sc} + \frac{1}{R_o}u \xrightarrow{R_o \neq \infty} u = R_o i_{sc} + R_o i = u_{oc} + R_o i \qquad (3.4.4)$$

Where $u_{oc} = R_o i_{sc}$.

As long as the resistance of the Norton equivalent circuit is not infinite (not the current source branch), it can be converted to the Thevenin equivalent circuit.

2) When the equivalent transformation of the Thevenin/ Norton equivalent circuit

(1) R_o takes the same value but is connected in a different way.

(2) The direction of the u_{oc} is opposite to the direction of the i_{sc}.

Another way to find the Thevenin/ Norton equivalent resistance R_o:

$$i_{sc} = \frac{u_{oc}}{R_o} \text{ or } u_{oc} = R_o i_{sc} \rightarrow R_o = \frac{u_{oc}}{i_{sc}}$$

There is no need to set all independent sources within the active resistor single-port N to zero to obtain the corresponding passive resistor single-port N_0 and then calculate the equivalent resistance. Instead, one can directly determine u_{oc} and i_{sc} within N, and the ratio of these two values yields the equivalent resistance R_o.

3.4.3 Application of Thevenin's Theorem and Norton's Theorem

Thevenin's theorem and Norton's theorem are primarily used for determining the voltage or current in a particular branch or the dynamic circuit of a single dynamic component within a resistive circuit. The resistive circuit outside the branch under consideration is represented as an active resistor single-port using Thevenin's equivalent circuit and Norton's equivalent circuit.

(1) The two steps for finding the Thevenin equivalent circuit and Norton equivalent circuit of an active resistor single port.

Firstly, calculate the open circuit voltage u_{oc} or short circuit current i_{sc} of N port, and then calculate the Thevenin/Norton equivalent resistance of N.

There are two methods to calculate the equivalent resistance R_o: one is the external power supply method, which adds a current source to the N_0 (all independent power sources within N are set to zero) port corresponding to N to calculate the port voltage, or adds a voltage source to the port to calculate the port current; The second is to simultaneously calculate the u_{oc} and i_{sc} of N, and the ratio of the two is R_o.

Example 3.6 Find the Thevenin equivalent circuit and Norton equivalent circuit of an active resistor single port as shown in Figure 3.4.6.

Solution: When calculating the open circuit voltage u_{oc} of port N ($I=0$, as shown in Figure 3.4.7), there is

$$U_{oc}=\frac{18}{12+6}\times 12-\left(\frac{6\times 12}{12+6}\right)\times 2=12-8=4(\text{V})$$

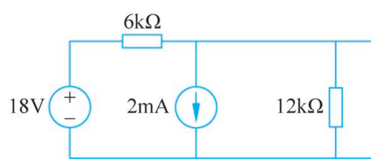


Figure 3.4.6 Example 3.6 circuit

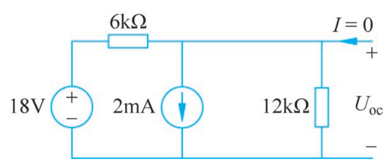


Figure 3.4.7 Example 3.6 diagram (1)

When calculating the short-circuit current I_{sc} of port N ($U=0$, as shown in Figure 3.4.8), there is

$$I_{sc}=\frac{18}{6}-2=3-2=1(\text{mA})$$

There are two methods for calculating the Thevenin equivalent resistance and Norton equivalent resistance R_o: one is the external power supply method (as shown in Figure 3.4.9), where N corresponds to N_0 (with an 18V voltage source and a 2mA current source set to zero within N) and a current source I is added to the port to calculate the port voltage U.

$$R_o=\frac{U}{I}=\frac{6\times 12}{12+6}=4(\mathrm{k\Omega})$$

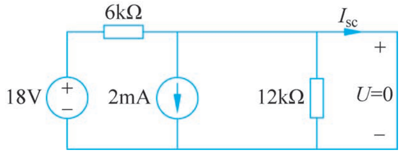


Figure 3.4.8 Example 3.6 diagram (2)

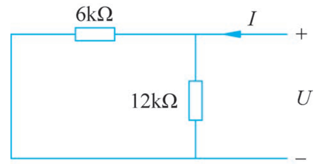


Figure 3.4.9 Example 3.6 diagram (3)

The second is the ratio of U_{oc} and I_{sc} of N (as shown in Figure 3.4.10):

$$R_o=\frac{U_{oc}}{I_{sc}}=\frac{4}{1}=4\mathrm{k\Omega}$$

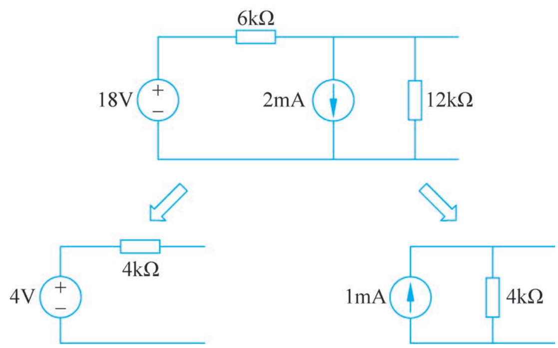


Figure 3.4.10 Example 3.6 diagram (4)

Example 3.7 In the active resistor single port shown in Figure 3.4.11, $u_S=0.05\cos t(\mathrm{V})$, calculate its Thevenin equivalent circuit.

Solution: As shown in Figure 3.4.12, when calculating the open circuit voltage u_{oc} of port N, $i=0$.

$$u_{R1}=2\times\frac{5-0.05\cos t}{2+3}=2-0.02\cos t(\mathrm{V})$$

$$u_{oc}=-4\times(2-0.02\cos t-1.5)\times 2=-4+0.16\cos t(\mathrm{V})$$

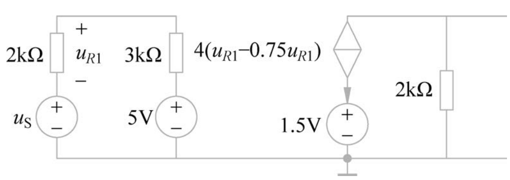


Figure 3.4.11 Example 3.7 circuit

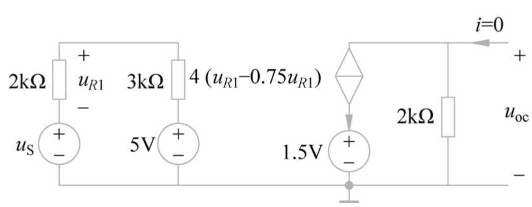


Figure 3.4.12 Example 3.7 diagram(1)

The external power supply method is used to find Thevenin equivalent resistance R_o, as shown in Figure 3.4.13. The u_S, 5V and 1.5V voltage sources are set to zero, and the current i is added to the N_0 port corresponding to N to find the voltage u.

$$u_{R1}=0$$

$$4u_{R1}=0$$

$$u=2i$$

$$R_o=\frac{u}{i}=2(\text{k}\Omega)$$

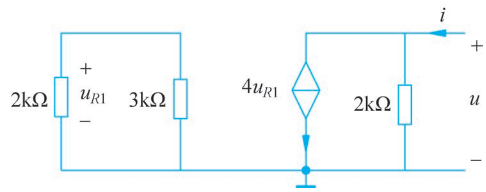


Figure 3.4.13 Example 3.7 diagram (2)

The final Thevenin equivalent circuit in Example 3.7 is shown in Figure 3.4.14.

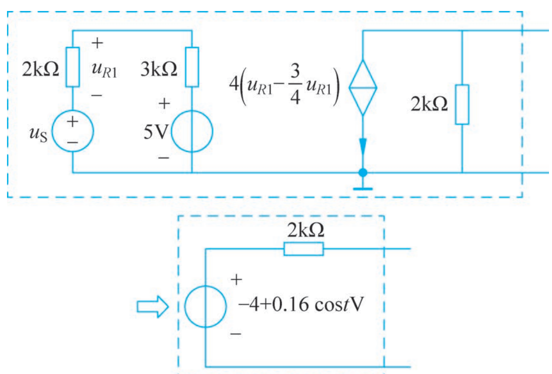


Figure 3.4.14 Final Thevenin equivalent circuit of Example 3.7

(2) Find the voltage or current of a branch in the resistance circuit.

For the resistive circuit outside the branch of interest, represent it as an active resistor single-port using Thevenin's equivalent circuit and Norton's equivalent circuit. Then, calculate the branch voltage or current for a single-loop circuit or a single independent node circuit (resistor voltage divider circuit or resistor current divider circuit). After converting to a single-loop circuit, there are specifically two forms:

① Resistor voltage divider circuit: a number of resistors and a voltage source composed of a single loop circuit. A resistor voltage divider circuit composed of two resistors and a voltage source, as shown in Figure 3.4.15.

Loop current:

$$u_S=u_1+u_2=R_1i+R_2i=(R_1+R_2)i$$

$$i=\frac{1}{R_1+R_2}u_S$$

Voltage division across each resistor:

$$u_1 = R_1 i = \frac{R_1}{R_1 + R_2} u_S, \quad u_2 = R_2 i = \frac{R_2}{R_1 + R_2} u_S$$

n resistors and a voltage source composed of resistance a resistor voltage divider circuit:

voltage division formula

$$u_i = \frac{R_i}{\sum_{j=1}^{n} R_j} u_S, \quad i = 1, 2, \cdots, n \tag{3.4.5}$$

② Resistor current divider circuit: A single independent node circuit composed of several resistors and a current source. A resistor current divider circuit composed of two resistors and a current source, as shown in Figure 3.4.16.

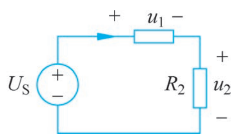


Figure 3.4.15 Resistor voltage divider circuit

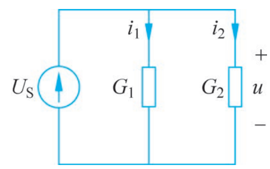


Figure 3.4.16 Resistor current divider circuit

Voltage at both ends of parallel resistor:

$$i_S = i_1 + i_2 = G_1 u + G_2 u = (G_1 + G_2) u$$

$$u = \frac{1}{G_1 + G_2} i_S$$

Current divided by each resistor:

$$i_1 = G_1 u = \frac{G_1}{G_1 + G_2} i_S = \frac{R_2}{R_1 + R_2} i_S$$

$$i_2 = G_2 u = \frac{G_2}{G_1 + G_2} i_S = \frac{R_1}{R_1 + R_2} i_S$$

Resistor current divider circuit composed of n resistors and a current source:

Diversion formula

$$i_i = \frac{G_i}{\sum_{j=1}^{n} G_j} i_S, \quad i = 1, 2, \cdots, n \tag{3.4.6}$$

Example 3.8 Calculate the current i of the bridge circuit shown in Figure 3.4.17. If i=0 (bridge balance) is required, what relationship should be satisfied between the bridge arm resistances?

Solution: When calculating the open circuit voltage u_{oc} of a single port N of an active resistor other than R_L, $i=0$, as shown in Figure 3.4.18.

$$u_{oc} = u_a - u_b = \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4}\right) u_S$$

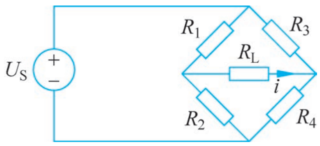


Figure 3.4.17 Example 3.8 circuit

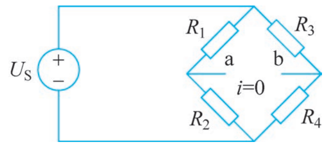


Figure 3.4.18 Example 3.8 diagram(1)

The method for calculating the equivalent resistance R_o of Thevenin is the external power supply method (as shown in Figure 3.4.19), where a current source i is added to the N_0 (u_S within N is set to zero) port corresponding to N, and the port voltage u is calculated.

For this example, the resistance R_o in the Thevenin equivalent circuit (as shown in Figure 3.4.20) is

$$R_o=(R_1 /\!/ R_2)+(R_3 /\!/ R_4)$$

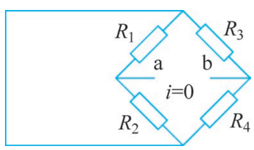


Figure 3.4.19 Example 3.8 diagram (2)

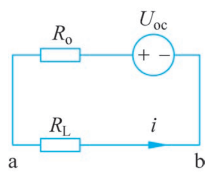


Figure 3.4.20 Example 3.8 diagram (3)

Find the i of the resistor voltage divider circuit:

$$i=\frac{u_{oc}}{R_o+R_L}=\frac{\left(\dfrac{R_2}{R_1+R_2}-\dfrac{R_4}{R_3+R_4}\right)u_S}{(R_1 /\!/ R_2)+(R_3 /\!/ R_4)+R_L}$$

When $i=0$, there is

$$\frac{R_2}{R_1+R_2}-\frac{R_4}{R_3+R_4}=0$$

$$\frac{R_2}{R_1+R_2}=\frac{R_4}{R_3+R_4}$$

$$R_2R_3=R_1R_4$$

Example 3.9 In the circuit shown in Figure 3.4.21, $u_S=0.05\cos t$ (V), calculate the voltage u_O.

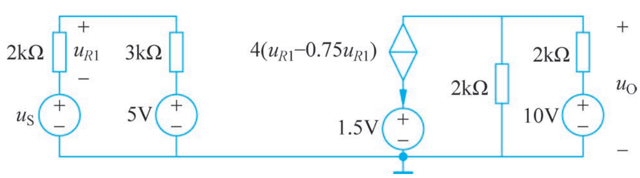


Figure 3.4.21 Example 3.9 circuit

Solution: To determine the Thevenin's equivalent circuit of the active resistor single-port N external to the series combination of a 2kΩ resistor and a 10V voltage source, as shown in Figure 3.4.22.

As shown in Figure 3.4.22, calculate the voltage u_O of a 2kΩ resistor and a 10V voltage source in series, as shown in Figure 3.4.23.

$$u_O = \frac{2}{2+2} \times (-14 + 0.16\cos t) + 10 = 3 + 0.08\cos t\,(\text{V})$$

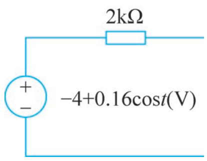


Figure 3.4.22 Example 3.9 diagram (1)

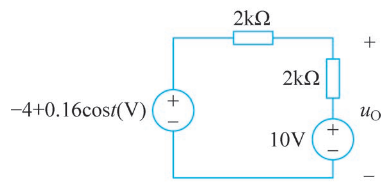


Figure 3.4.23 Example 3.9 diagram (2)

(3) To determine the voltage or current in the dynamic branch of a single dynamic component's dynamic circuit, represent the resistive circuit outside the dynamic branch under consideration as an active resistor single-port using Thevenin's equivalent circuit and Norton's equivalent circuit. Then, formulate and solve the differential equations for a single-loop circuit or a single independent node circuit to find the dynamic branch voltage or current.

Example 3.10 In the circuit shown in Figure 3.4.24, when the capacitor is inserted in the circuit at $t=0$, meanwhile $u(0)=0$, calculate the voltage of the capacitor when $t \geqslant 0$.

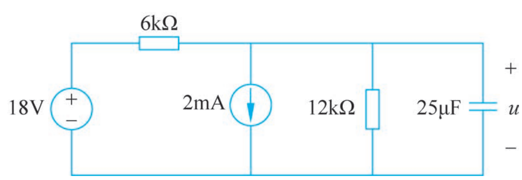


Figure 3.4.24 Example 3.10 circuit

Solution: Determine the Norton equivalent circuit for a single port N with an active resistance external to a 25μF capacitor, as shown in Figure 3.4.25. Determine the voltage u for the circuit with a single independent node, as shown in Figure 3.4.26.

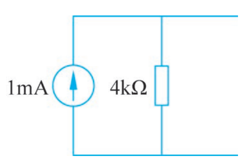


Figure 3.4.25 Example 3.10 Norton equivalent circuit

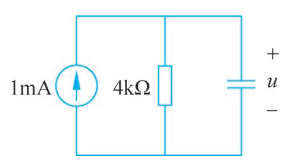


Figure 3.4.26 Example 3.10 single independent node circuit

According to the nodal analysis KCL, the following equations are derived:

$$25\times10^{-6}\frac{du}{dt}+\frac{1}{4\times10^{3}}u=1\times10^{-3},\quad t\geqslant0$$

$$u(0)=0$$

$$u=Ke^{-10t}+4(\mathrm{V}),\quad t\geqslant0$$

$$u(0)=0$$

$$u=4(1-e^{-10t})(\mathrm{V}),\quad t\geqslant0$$

3.5 Nodal Analysis Method

3.5.1 Node Voltage

The circuit example is shown in Figure 3.5.1.

In Figure 3.5.1, the number of branches $b=5$, and the branch voltages are $u_1, u_2, u_3, u_{i_{S1}}, u_{i_{S2}}$. The number of independent nodes is $n-1=2$, which is less than the number of branches $b=5$.

Choose any node as the reference node, also known as the ground node. After selecting the reference node, the voltages of the remaining independent nodes with respect to the reference node are the node voltages, denoted as u_a and u_b, respectively.

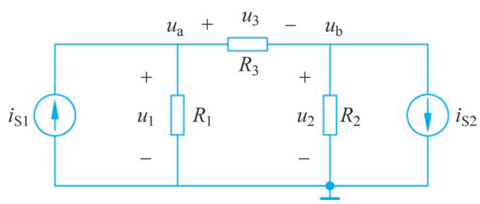


Figure 3.5.1 Circuit diagram illustrating node voltages

The KVL equation with node voltages as variables is as follows:

$$-u_a+(u_a-u_b)+u_b=0$$

—KVL does not impose linear constraints on node voltages:

$$u_a+u_{i_{S1}}=0$$

$$-u_b+u_{i_{S2}}=0$$

Combining the VCR with the KCL equation where node voltages are the variables, the equation is as follows:

$$G_1u_a+G_3(u_a-u_b)-i_{S1}=(G_1+G_3)u_a-G_3u_b-i_{S1}=0$$

$$G_2u_b-G_3(u_a-u_b)+i_{S2}=-G_2u_a+(G_2+G_3)u_b+i_{S2}=0$$

The relationship between branch voltages and node voltages is as follows:

$$u_1=u_a$$

$$u_2=u_b$$

$$u_3=u_a-u_b$$

$$u_{i_{S1}} = -u_a$$

$$u_{i_{S2}} = u_b$$

Node voltages possess the following characteristics:

(1) Independence—KVL does not impose linear constraints on node voltages.

(2) Solvability—There are $n-1$ node voltages and $n-1$ equations combining VCR and KCL with node voltages as variables.

(3) Completeness—All branch voltages are linear combinations of node voltages.

From the above analysis, it can be seen that the node analysis method takes the node voltages as variables, formulates $n-1$ KCL equations combined with VCR, and solves for $n-1$ node voltages. After solving, the voltages and currents of the b branches can be determined.

3.5.2 Writing the Node Equation

Example circuit as shown in Figure 3.5.2.

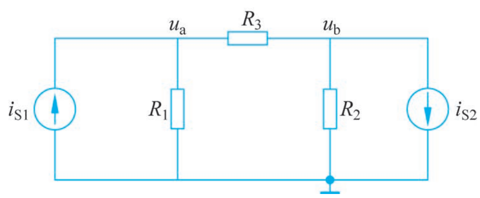


Figure 3.5.2 Circuit diagram illustrating node equations

Combining the VCR with node voltages as variables, the KCL equation is listed as follows:

$$G_1 u_a + G_3(u_a - u_b) - i_{S1} = (G_1 + G_3)u_a - G_3 u_b - i_{S1} = 0$$

$$G_2 u_b - G_3(u_a - u_b) + i_{S2} = -G_2 u_a + (G_2 + G_3)u_b + i_{S2} = 0$$

$$(G_1 + G_3)u_a - G_3 u_b = i_{S1}$$

$$-G_2 u_a + (G_2 + G_3)u_b = -i_{S2}$$

The self-conductances G_{aa} and G_{bb} of nodes a and b are the sum of the conductances of the branches connected to nodes a and b, respectively. Therefore, $G_{aa} = G_1 + G_3$, $G_{bb} = G_2 + G_3$, and

$$(G_1 + G_3)u_a - G_3 u_b = i_{S1}$$

$$-G_2 u_a + (G_2 + G_3)u_b = -i_{S2}$$

The mutual conductance between nodes, $G_{ab} = G_{ba}$, is the sum of the conductances of all branches that are connected to both nodes a and b. Thus, $G_{ab} = G_{ba} = G_3$.

The sum of the currents flowing into nodes a and b, i_{Saa} and i_{Sbb}, respectively, $i_{Saa} = i_{S1}$, $i_{Sbb} = -i_{S2}$.

(1) Node equations for a resistive circuit without voltage sources and controlled

sources are formulated. For a circuit with b branches and n nodes, there are $n-1$ node equations:

$$\sum_{j=1}^{n-1} \pm G_{ij} u_j = i_{Sii}, \quad i = 1, 2, \cdots, n-1 \qquad (3.5.1)$$

When $j = i$, G_{ij} represents the self-conductance of the node i. When $j \neq i$, G_{ij} represents the mutual conductance between nodes i and j. i_{Sii} represents the sum of the current sources flowing into node i. The positive or negative sign in front of the conductance depends on whether it is a self-conductance or a mutual conductance. For self-conductance, it is taken as positive, and for mutual conductance, it is taken as negative.

Example 3.11 Calculate the power of the current source in the circuit shown in Figure 3.5.3.

Solution: Assuming a reference node and node voltages U_1, U_2, and U_3 as shown in Figure 3.5.4.

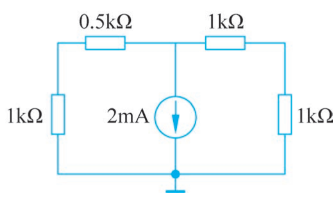


Figure 3.5.3 Circuit for Example 3.11

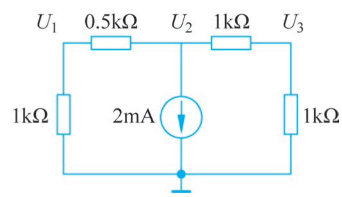


Figure 3.5.4 Solution diagram for Example 3.11

Based on the node voltages, the node equations are acquired using (KCL). This results in the following system of equations:

$$\begin{cases} (1+2)U_1 - 2U_2 = 3U_1 - 2U_2 = 0 \\ -2U_1 + (2+1)U_2 - U_3 = -2U_1 + 3U_2 - U_3 = -2 \\ -U_2 + (1+1)U_3 = -U_2 + 2U_3 = 0 \end{cases}$$

By solving the system of equations, solutions founded are:

$$\begin{cases} U_1 = -\dfrac{8}{7}\text{V} \\ U_2 = -\dfrac{12}{7}\text{V} \\ U_3 = -\dfrac{6}{7}\text{V} \\ P = 2U_2 = 2(-12/7) = -\dfrac{24}{7}(\text{mW}) \end{cases}$$

(2) Formulation of node equations for a resistive circuit containing voltage sources but without controlled sources.

If a voltage source is only connected to one node, the voltage source determines the voltage at that node, and it's unnecessary to write the node equation for that node. When a voltage source is associated with two nodes, it imposes a constraint on the voltage between these nodes (supplementary equation). When formulating the node equation for such a node, consider the current through the voltage source as an unknown, treating the voltage source as a current source for this unknown current; the rest is formulated similarly to the node equations for a resistive circuit without voltage sources and controlled sources.

Example 3.12 Calculate the current I in the circuit shown in Figure 3.5.5.

Solution: Assuming a reference node and node voltages U_1, U_2, and U_3 as shown in Figure 3.5.6.

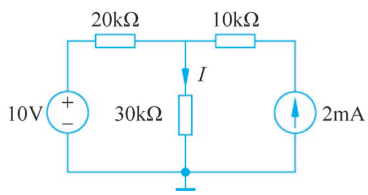


Figure 3.5.5 Circuit for Example 3.12

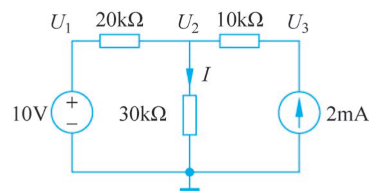


Figure 3.5.6 Solution diagram for Example 3.12

Determine the voltage at node 1:

$$U_1 = 10\text{V}$$

The node equations for node 2 and node 3 are as follows:

$$-\frac{1}{20}U_1 + \left(\frac{1}{20}+\frac{1}{30}+\frac{1}{10}\right)U_2 - \frac{1}{10}U_3 = -\frac{1}{20}\times 10 + \frac{11}{60}U_2 - \frac{1}{10}U_3 = 0$$

$$-\frac{1}{10}U_2 + \frac{1}{10}U_3 = 2$$

Solution obtained:

$$11U_2 - 6U_3 = 30$$

$$-U_2 + U_3 = 20$$

$$U_2 = 30\text{V}$$

$$U_3 = 50\text{V}$$

$$I = U_2/30 = 30/30 = 1(\text{mA})$$

Example 3.13 Calculate the voltage U in the circuit shown in Figure 3.5.7.

Solution: Assuming a reference node and node voltages U_1, U_2, and an unknown current I for the voltage source, as shown in Figure 3.5.8.

The constraint relationship between the voltage at node 1 and the voltage at node 2 (supplementary equation): $U_1 - U_2 = 3\text{V}$

The node equations for node 1 and node 2 are as follows:

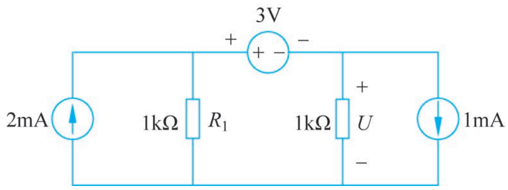


Figure 3.5.7 Circuit for Example 3.13

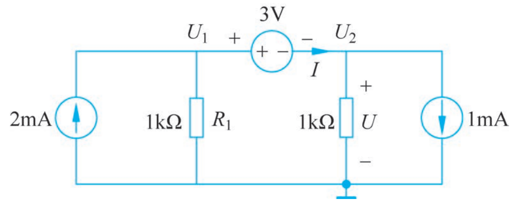


Figure 3.5.8 Solution diagram for Example 3.13

$$U_1 = 2 - I$$
$$U_2 = I - 1$$
$$U_1 + U_2 = 1$$
$$U_1 - U_2 = 3$$
$$U_1 = 2\text{V}$$
$$U_2 = -1\text{V}$$
$$I = 0$$
$$U = U_2 = -1\text{V}$$

An alternative solution for Example 3.13:

Assuming a reference node and node voltages U_1 and U_2 as shown in Figure 3.5.9.

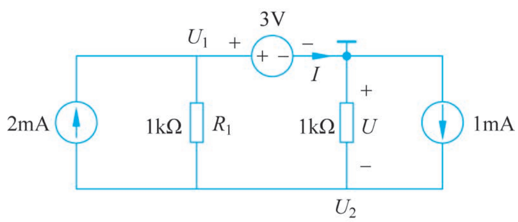


Figure 3.5.9 Solution diagram for Example 3.13(2)

Determine the voltage at node 1:

$$U_1 = 3\text{V}$$

The node equation for node 2:

$$-U_1 + (1+1)U_2 = -U_1 + 2U_2 = -3 + 2U_2 = 1 - 2 = -1$$
$$U_2 = 1\text{V}$$
$$U = -U_2 = -1\text{V}$$

(3) Formulation of node equations for circuits with controlled sources and resistors.

Convert the control variable of the controlled source into node voltages, treat the controlled source as an independent source in the equation formulation, and then rearrange the terms by moving them to the appropriate side of the equation. This process is otherwise identical to the formulation of node equations for circuits without controlled sources.

Example 3.14 Calculate the current I in the circuit shown in Figure 3.5.10.

Solution: Assuming a reference node and node voltages U_1, U_2, U_3 as shown in Figure 3.5.11.

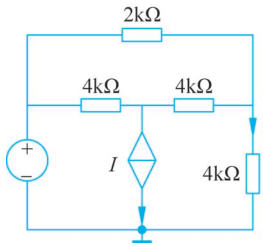


Figure 3.5.10 Circuit for Example 3.14

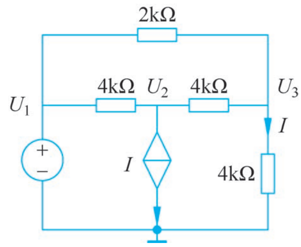


Figure 3.5.11 Solution diagram for Example 3.14

Determine the voltage at node 1:

$$U_1 = 2\text{V}$$

The controlling quantity of the controlled source can be converted to $U_3/4$.

The node equations for node 2 and node 3 are as follows:

$$-\frac{1}{4}U_1 + \left(\frac{1}{4}+\frac{1}{4}\right)U_2 - \frac{1}{4}U_3 = -\frac{1}{4}\times 2 + \frac{1}{2}U_2 - \frac{1}{4}U_3 = -I = -\frac{U_3}{4}$$

$$-\frac{1}{2}U_1 - \frac{1}{4}U_2 + \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right)U_3 = -\frac{1}{2}\times 2 - \frac{1}{4}U_2 + U_3 = 0$$

Solution obtained:

$$U_2 = 1\text{V}$$
$$-U_2 + 4U_3 = 4\text{V}$$
$$U_1 = 2\text{V}$$
$$U_2 = 1\text{V}$$
$$U_3 = 1.25\text{V}$$
$$I = U_3/4 = 1.25/4 = 0.3125(\text{mA})$$

*3.5.3 Series *RC* Circuit with A Step Input

The series RC circuit with a step input is shown in Figure 3.5.12.

Assume that the waveform u_S is a step voltage with an amplitude of U, which is applied to the circuit at $t=0$. However, this time, assume that the capacitor has a voltage of U_0 before the step, meaning that the initial condition of the circuit is as follows:

$$u_C = U_0 \tag{3.5.2}$$

Using the nodal analysis method, we can obtain the differential equation. Applying KCL to the node with voltage u_C, we have:

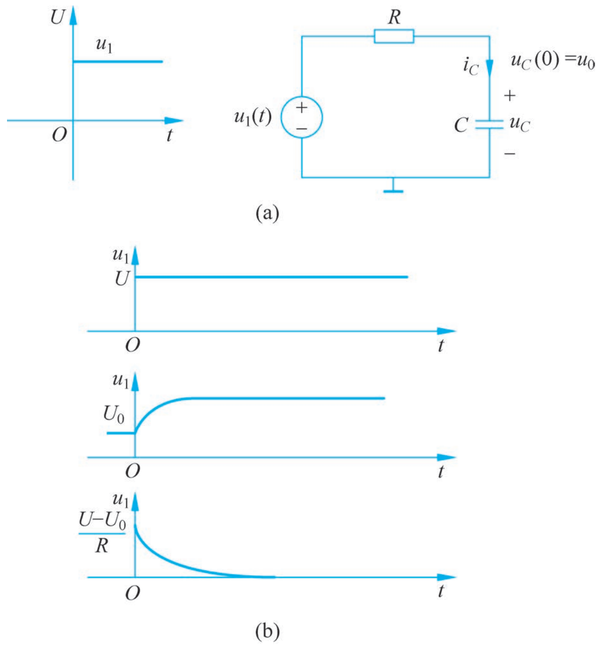


Figure 3.5.12 Series *RC* circuit with a step input

$$\frac{u_C - u_1}{R} + C\frac{du_C}{dt} = 0 \tag{3.5.3}$$

Dividing both sides of the equation by C and rearranging, we get:

$$\frac{du_C}{dt} + \frac{u_C}{RC} = \frac{u_1}{RC} \tag{3.5.4}$$

The homogeneous equation is:

$$\frac{du_{Ch}}{dt} + \frac{u_{Ch}}{RC} = 0 \tag{3.5.5}$$

As expected, this equation is the same as the Equation (3.2.7) that represents the Norton equivalent circuit, because the Norton equivalent circuit and the Thevenin equivalent circuit are equivalent. By borrowing the homogeneous solution from Equation (3.2.7), we have:

$$u_{Ch} = Ae^{-t/RC} \tag{3.5.6}$$

In the equation, RC represents the time constant of the circuit.

Then find the particular solution. Since the input is a step signal with an amplitude of U, the particular solution satisfies:

$$\frac{du_{Cp}}{dt} + \frac{u_{Cp}}{RC} = \frac{U}{RC} \tag{3.5.7}$$

Since the power source is a step function and becomes a constant at large t, let's assume that the particular solution has the form:

$$u_{Cp} = K \tag{3.5.8}$$

Substituting Equation (3.5.8) into Equation (3.5.7):

$$\frac{K}{RC} = \frac{U}{RC} \tag{3.5.9}$$

This indicates that $K = U$. Therefore, the particular solution is:

$$u_{Cp} = U \tag{3.5.10}$$

Adding u_{Ch} and u_{Cp} together, we obtain the complete solution as:

$$u_C = U + A\mathrm{e}^{-t/RC} \tag{3.5.11}$$

Now we can use the initial condition to determine A. Since the voltage across the capacitor must be continuous at $t=0$, we have:

$$u_C(t=0) = U_0 \tag{3.5.12}$$

Therefore, at $t=0$, we can deduce from Equation (3.5.11) that:

$$A = U_0 - U \tag{3.5.13}$$

The complete solution for the voltage across the capacitor for $t>0$ is:

$$u_C = U + (U_0 - U)\mathrm{e}^{-t/RC} \tag{3.5.14}$$

In the equation, U represents the input driving voltage for $t>0$, and U_0 represents the initial voltage across the capacitor.

Let's do a quick verification: Substitute $t = 0$, obtain solution $u_C(0) = U_0$; Substitute $t = \infty$, obtain solution $u_C(\infty) = U$. Both boundary conditions are as expected, with the initial value of the capacitor voltage being U_0, and after a long period of time, the voltage from the source will be fully applied across the capacitor.

By rearranging the terms in Equation (3.5.14), the following equivalent form is obtained:

$$u_C = U_0\mathrm{e}^{-t/RC} + U(1-\mathrm{e}^{-t/RC}) \tag{3.5.15}$$

The current flowing through the capacitor is given by:

$$i_C = C\frac{\mathrm{d}u_C}{\mathrm{d}t} = \frac{U-U_0}{R}\mathrm{e}^{-t/RC} \tag{3.5.16}$$

The expression for i_C also matches the expectation, as when t is very large, i_C will be zero. At $t=0$, the capacitor acts like a voltage source with a voltage of U_0, so the current at $t=0$ will be $(U-U_0)/R$.

These waveforms are shown in Figure 3.5.12(b).

If we want to find the voltage across the resistor, u_R, we can easily apply KVL to obtain:

$$u_R = u_1 - u_C \tag{3.5.17}$$

Where the input terminal of the resistor is taken as the positive reference direction for u_R.

Alternatively, taking the product of the current and the resistance will also give the voltage across the resistor u_R as:

$$u_R = i_C R \tag{3.5.18}$$

Equation (3.5.14) was derived under the assumption that the initial condition (U_0) and the input (step signal U) are both non-zero.

Substituting $U=0$ into Equation (3.5.14), obtain the equation:

$$u_C = U_0 e^{-t/RC} \tag{3.5.19}$$

Substituting $U_0=0$ into Equation (3.5.14), obtain the equation:

$$u_C = U - U e^{-t/RC} \tag{3.5.20}$$

The total response is the sum of the two equations, by adding the right sides of Equation (3.5.19) and Equation (3.5.20), and comparing it with the right side of Equation (3.5.14), then are able to prove this theory.

*3.5.4 Series *RC* Circuit with Square Wave Input

The study of the waveforms in Figures 3.2.3(a) and 3.2.3(b) shows that the presence of the capacitor changes the shape of the input square wave. When a square wave pulse is applied to an *RC* circuit, the resulting pulse is not a square wave; it rises and falls slowly. The capacitor allows the circuit to perform certain waveform shaping. This concept can be further established through experiments with square wave driving.

In this experiment, the Thevenin equivalent circuit shown in Figure 3.5.13 is used. The power supply can be a standard laboratory square wave generator. The input square wave is labeled as 1 in Figure 3.5.13. Depending on the relationship between the period of the driving square wave and the time constant *RC* of the network, various distinct waveforms of $u_C(t)$ can be obtained. These waveforms are variations of the solutions obtained earlier.

When the time constant of the circuit is much shorter compared to the period of the square wave, the exponential decay occurs relatively faster, as shown in waveform 2 in Figure 3.5.13. The capacitor waveform closely resembles the input waveform, except for some small rounding at the corners.

If the time constant occupies a significant portion of the pulse duration, the waveform of the solution will be as shown in waveform 3 in Figure 3.5.13. Note that the graph indicates that the transient process is still almost nearing its end. Therefore, to apply this solution, the product of *RC* must have an upper limit. As mentioned above, assuming that the simple transient process ends after a time greater than 5 times the time constant, the product of *RC* must be less than 1/5 of the pulse length or 1/10 of the square wave period to apply this solution.

When the time constant of the circuit is much larger than the period of the square wave, the resulting waveform is as shown in waveform 4 in Figure 3.5.13. In this

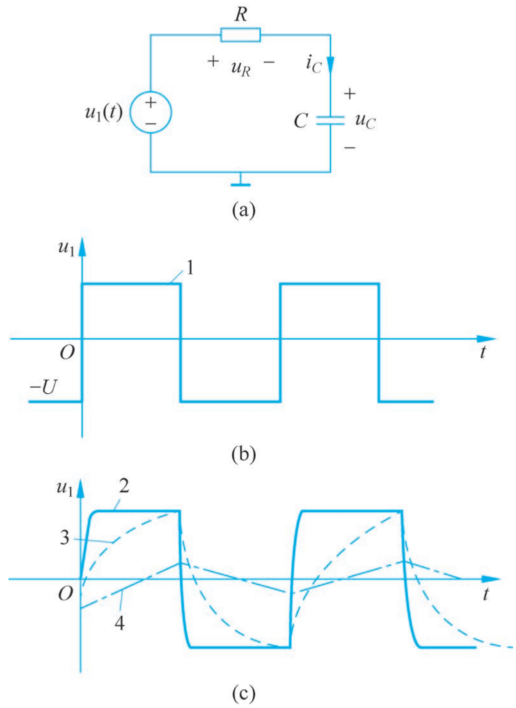


Figure 3.5.13 Response of series *RC* circuit to square wave input

case, the transient process clearly has not ended. In fact, only the first part of the exponential function is observed. The waveform appears like a triangular wave, which is the integral of the input waveform. This can be seen from the differential equation describing the circuit. Applying KVL yields:

$$u_1 = i_C R + u_C \tag{3.5.21}$$

Utilize the voltage-current relationship of the capacitor to derive the differential equation:

$$u_1 = RC\frac{\mathrm{d}u_C}{\mathrm{d}t} + u_C \tag{3.5.22}$$

It is evident that, based on Equation (3.5.22) or Figure 3.5.13, as the time constant of the circuit increases, the voltage across the capacitor, u_C, will decrease. In the case of waveform 4, where the time constant RC is significantly large, $u_C \ll u_1$. Hence, in this scenario, Equation (3.5.21) can be approximately expressed as:

$$u_1 \approx i_C R \tag{3.5.23}$$

From a physical standpoint, the current now only depends on the driving voltage and the resistance, as the capacitor voltage is nearly zero. Assuming u_C can be neglected, integrating both sides of Equation (3.5.22):

$$u_C \approx \frac{1}{RC}\int u_1 \mathrm{d}t + K \tag{3.5.24}$$

In the equation, the integration constant K is equal to zero. Therefore, when RC is large, the voltage across the capacitor is approximately the integral of the input voltage. This is a very useful signal processing property.

It is straightforward to calculate the voltage across the resistor in the circuit shown in Figure 3.5.13(a) (as the current can be obtained from the voltage across the capacitor):

$$u_R = i_C R = RC\frac{\mathrm{d}u_C}{\mathrm{d}t} \tag{3.5.25}$$

Considering the charging time period as an example and assuming that the transient process has already concluded, we can derive from Equation (3.2.22):

$$u_C = U(1 - \mathrm{e}^{-t/RC}) \tag{3.5.26}$$

Therefore

$$u_R = U\mathrm{e}^{-t/RC} \tag{3.5.27}$$

If the average value of the input signal u_1 is zero, meaning that u_1 varies between $-U/2$ and $+U/2$, the waveforms in Figure 3.5.8 will undergo minimal changes. Specifically, the average value of u_C will also be zero. When the transient process has concluded, as depicted in waveforms 2 and 3 in Figure 3.5.14, the offset will be $-U/2$ and $+U/2$, respectively.

3.6 Phasor Model for Sinusoidal Steady-State Circuits

3.6.1 Dynamic Circuits Driven by Sinusoidal Signals

Example 3.15 Consider the circuit depicted in Figure 3.6.1, where the current source $i_S = 5\cos(10t + 45°)$(mA). The switch is closed at $t = 0$, and it is known that $u_C(0) = 0$. Determine the values of u_C, u_R, u_S, i_C and i_R for sufficiently large values of t.

Solution: Based on the application of KCL at the node, we can deduce:

$$0.3\frac{\mathrm{d}u_C}{\mathrm{d}t} + 4u_C = 5\cos(10t + 45°), \quad t \geqslant 0$$

$$u_C(0) = 0$$

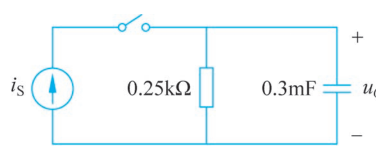


Figure 3.6.1 Sinusoidal excitation dynamic circuits

Solving the differential equation yields the solution:

$$u_C = u_{Ch} + u_{Cp} = K\mathrm{e}^{-\frac{40t}{3}} + u_{Cp}(\mathrm{V}), \quad t \geqslant 0$$

$$u_C(0) = 0$$

Assume $u_{Cp} = U_{Cm}\cos(10t + \varphi_u) \rightarrow$

$$-10 \times 0.3U_{Cm}\sin(10t + \varphi_u) + 4U_{Cm}\cos(10t + \varphi_u) = 5\cos(10t + 45°)$$

$$-\frac{3}{\sqrt{3^2 + 4^2}}U_{Cm}\sin(10t + \varphi_u) + \frac{4}{\sqrt{3^2 + 4^2}}U_{Cm}\cos(10t + \varphi_u) = \frac{5}{\sqrt{3^2 + 4^2}}\cos(10t + 45°)$$

Since $\varphi = \arctan\frac{3}{4} = 36.9°$, the solution:

$$-U_{Cm}\sin(10t+\varphi_u)\sin 36.9° + U_{Cm}\cos(10t+\varphi_u)\cos 36.9°$$
$$= U_{Cm}\cos(10t+\varphi_u+36.9°) = \cos(10t+45°)$$
$$U_{Cm} = 1,\quad \varphi_u = 45° - 36.9° = 8.1°$$

And

$$u_{Cp} = \cos(10t+8.1°)$$
$$u_C = K e^{-\frac{40t}{3}} + \cos(10t+8.1°)(\text{V}),\quad t \geqslant 0,\quad u_C(0) = 0$$
$$u_C = \cos(10t+8.1°) - 0.99e^{-\frac{40t}{3}}(\text{V}),\quad t \geqslant 0$$

For sufficiently large values of t

$$u_C = \cos(10t+8.1°)(\text{V})$$
$$u_R = u_S = \cos(10t+8.1°)(\text{V})$$
$$i_C = 0.3\frac{\text{d}}{\text{d}t}\cos(10t+8.1°) = -3\sin(10t+8.1°) = 3\cos(10t+98.1°)(\text{mA})$$
$$i_R = \frac{\cos(10t+8.1°)}{0.25} = 4\cos(10t+8.1°)(\text{mA})$$

3.6.2 Sinusoidal Steady-State Circuits

Sinusoidal steady-state circuits refer to dynamic circuits that exhibit a steady response (response at sufficiently large values of t) under sinusoidal excitation. In sinusoidal steady-state circuits, all branch voltages and currents are sinusoidal quantities with the same frequency as the excitation signal.

3.6.3 Phasor Representation of Sinusoidal Quantities

1. Phasor representation of the amplitude of a sinusoidal quantity using Euler's formula

$$e^{j(\omega t+\varphi)} = \cos(\omega t+\varphi) + j\sin(\omega t+\varphi) \tag{3.6.1}$$
$$\cos(\omega t+\varphi) = \text{Re}[e^{j(\omega t+\varphi)}]\sin(\omega t+\varphi) = \text{Im}[e^{j(\omega t+\varphi)}] \tag{3.6.2}$$
$$u = U_m\cos(\omega t+\varphi_u) = \text{Re}[U_m e^{j(\omega t+\varphi_u)}] = \text{Re}[\dot{U}_m e^{j\omega t}]$$
$$i = I_m\cos(\omega t+\varphi_i) = \text{Re}[I_m e^{j(\omega t+\varphi_i)}] = \text{Re}[\dot{I}_m e^{j\omega t}] \tag{3.6.3}$$

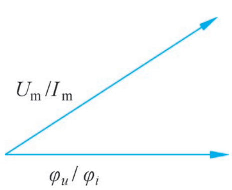


Figure 3.6.2 Vector graph

The phasor diagram is shown in Figure 3.6.2.

The voltage phasor is:

$$\dot{U}_m = U_m e^{j\varphi_u} = U_m\angle\varphi_u$$

The current phasor is:

$$\dot{I}_m = I_m e^{j\varphi_i} = I_m\angle\varphi_i$$

2. The effective value and phasor of sinusoidal quantities

Compare the energy consumed by a sinusoidal current passing through the same resistor in one period with the energy consumed by a direct current in the same duration. The effective value of the sinusoidal quantity is the root mean square (RMS) value, and from the perspective of energy consumption, the two currents are equivalent, that is, $W_i = W_I$.

$$\begin{aligned} I &= \sqrt{\frac{1}{T}\int_0^T i^2(t)\,\mathrm{d}t} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos(\omega t + \varphi_i)^2\,\mathrm{d}t} \\ &= \sqrt{\frac{1}{T}\int_0^T I_m^2\,\frac{1}{2}[1+\cos(2\omega t + 2\varphi_i)]\mathrm{d}t} \\ &= \frac{1}{\sqrt{2}} I_m = 0.707 I_m \end{aligned} \tag{3.6.4}$$

The RMS phasor of a sinusoidal quantity:

$$\begin{cases} u = U_m\cos(\omega t+\varphi_u) = \sqrt{2}U\cos(\omega t+\varphi_u) = \mathrm{Re}[\sqrt{2}U\mathrm{e}^{\mathrm{j}(\omega t+\varphi_u)}] = \sqrt{2}\,\mathrm{Re}[\dot{U}\mathrm{e}^{\mathrm{j}\omega t}] \\ i = I_m\cos(\omega t+\varphi_i) = \sqrt{2}I\cos(\omega t+\varphi_i) = \mathrm{Re}[\sqrt{2}I\mathrm{e}^{\mathrm{j}(\omega t+\varphi_i)}] = \sqrt{2}\,\mathrm{Re}[\dot{I}\mathrm{e}^{\mathrm{j}\omega t}] \end{cases} \tag{3.6.5}$$

The RMS phasor of the voltage is $\dot{U} = U\mathrm{e}^{\mathrm{j}\varphi_u} = U\angle\varphi_u$

The RMS phasor of the current is $\dot{I} = I\mathrm{e}^{\mathrm{j}\varphi_i} = I\angle\varphi_i$

$$\dot{U} = \frac{1}{\sqrt{2}}\dot{U}_m, \quad \dot{I} = \frac{1}{\sqrt{2}}\dot{I}_m$$

The conversion of a sinusoidal quantity to a phasor, and from a phasor and angular frequency to a sinusoidal quantity are as follows:

$$i = 5\cos(314t + 60°)(\mathrm{mA}) \rightarrow \dot{I}_m = 5\angle 60°(\mathrm{mA}) \tag{3.6.6}$$

$$\dot{U} = -5\angle -30° = 5\angle 150°(\mathrm{V}) \quad \text{and} \quad \omega = 2\pi(\mathrm{rad/s})$$

$$u = 5\sqrt{2}\cos(2\pi t + 150°)(\mathrm{V})$$

3.6.4 Phasor Calculation of Sinusoidal Quantities

Sinusoidal quantities and phasors exhibit the following three properties:

(1) Uniqueness. Let the corresponding phasors be: $i_1 \leftrightarrow \dot{I}_1, i_2 \leftrightarrow \dot{I}_2$. If $i_1 = i_2$, then $\dot{I}_1 = \dot{I}_2$;

(2) Linearity. Let $i_1 \leftrightarrow \dot{I}_1, \cdots, i_n \leftrightarrow \dot{I}_n$, there are $\alpha_1 i_1 + \cdots + \alpha_n i_n \leftrightarrow \alpha_1\dot{I}_1 + \cdots + \alpha_n\dot{I}_n$;

(3) Differentiation. Let $i \leftrightarrow \dot{I}$, there are $\frac{\mathrm{d}i}{\mathrm{d}t} \leftrightarrow \mathrm{j}\omega\dot{I}, \cdots, \frac{\mathrm{d}^n i}{\mathrm{d}t^n} \leftrightarrow (\mathrm{j}\omega)^n\dot{I}$

$$\frac{\mathrm{d}i}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}[\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]=\mathrm{Re}[\frac{\mathrm{d}}{\mathrm{d}t}\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]=\mathrm{Re}[\mathrm{j}\omega\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]\leftrightarrow\mathrm{j}\omega\dot{I} \tag{3.6.7}$$

Example 3.16 Given $i_1=\sin(2t-30°)(\mathrm{mA})$, $i_2=\cos(2t+45°)(\mathrm{mA})$ find $\frac{\mathrm{d}i_1}{\mathrm{d}t}+2i_2$.

Solution: First, the current i_1, i_2 are expressed as the phasor, as shown in Figure 3.6.3.

$$i_1=\sin(2t-30°)=\cos(2t-120°)\leftrightarrow\dot{I}_{1\mathrm{m}}=1\angle-120°$$

$$i_2=\cos(2t+45°)\leftrightarrow\dot{I}_{2\mathrm{m}}=1\angle45°$$

$$\begin{aligned}\mathrm{j}2\dot{I}_{1\mathrm{m}}+2\dot{I}_{2\mathrm{m}}&=2\angle90°\times1\angle-120°+2\times1\angle45°=2\angle-30°+2\angle45°\\&=(1.732-\mathrm{j}1)+(1.414+\mathrm{j}1.414)=3.146+\mathrm{j}0.414\\&=3.17\angle7.5°\end{aligned}$$

$$\frac{\mathrm{d}i_1}{\mathrm{d}t}+2i_2=3.17\cos(2t+7.5°)(\mathrm{mA})$$

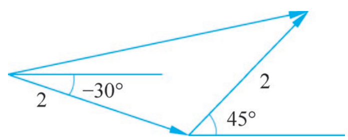


Figure 3.6.3 Example 3.16 diagram

3.6.5 Phasor Model of Sinusoidal Steady-State Circuit

1. Phasor model of resistor

For any resistor in a sinusoidal steady-state circuit, the VCR equation $\dot{U}=R\dot{I}$ or $\dot{U}_\mathrm{m}=R\dot{I}_\mathrm{m}$ is satisfied between the voltage phasor and the current phasor in the associated reference direction:

$$\begin{aligned}u&=\sqrt{2}U\cos(\omega t+\varphi_u)=\sqrt{2}\mathrm{Re}[\dot{U}\mathrm{e}^{\mathrm{j}\omega t}]=Ri=R\sqrt{2}I\cos(\omega t+\varphi_i)\\&=R\sqrt{2}\mathrm{Re}[\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]=\sqrt{2}\mathrm{Re}[R\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]\end{aligned}$$

$$\dot{U}=R\dot{I} \tag{3.6.8}$$

$$\dot{U}=U\angle\varphi_u=R\dot{I}=RI\angle\varphi_i \tag{3.6.9}$$

The effective value or amplitude meets $U=RI$ or $U_\mathrm{m}=RI_\mathrm{m}$, and the voltage phase is in phase with the current phase, $\varphi_u=\varphi_i$. As shown in Figure 3.6.4.

Figure 3.6.4 Vector diagram

2. Phasor model of inductor

For any inductor in a sinusoidal steady-state circuit, the VCR equation $\dot{U}=\mathrm{j}\omega L\dot{I}$ or $\dot{U}_\mathrm{m}=\mathrm{j}\omega L\dot{I}_\mathrm{m}$ is satisfied between the voltage phasor and the current phasor in the associated reference direction:

$$u=\sqrt{2}U\cos(\omega t+\varphi_u)=\sqrt{2}\,\mathrm{Re}[\dot{U}\mathrm{e}^{\mathrm{j}\omega t}]$$
$$=L\frac{\mathrm{d}i}{\mathrm{d}t}=L\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{2}I\cos(\omega t+\varphi_i)=L\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{2}\,\mathrm{Re}[\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]=\sqrt{2}\,\mathrm{Re}[\mathrm{j}\omega L\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]\rightarrow\dot{U}=\mathrm{j}\omega L\dot{I} \tag{3.6.10}$$

$$\dot{U}=U\angle\varphi_u=\mathrm{j}\omega L\dot{I}=\omega L\angle 90^\circ I\angle\varphi_i=\omega LI\angle(\varphi_i+90^\circ) \tag{3.6.11}$$

The effective value or amplitude meets $U=\omega LI$ or $U_m=\omega LI_m$, and the voltage phase is 90° ahead of the current phase, $\varphi_u=\varphi_i+90°$. As shown in Figure 3.6.5.

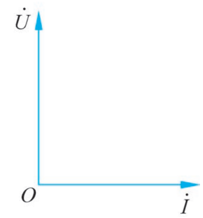


Figure 3.6.5 Vector diagram

3. Phasor model of capacitor

For any capacitor in a sinusoidal steady-state circuit, the VCR equation $\dot{U}=\frac{1}{\mathrm{j}\omega C}\dot{I}$ or $\dot{U}_m=\frac{1}{\mathrm{j}\omega C}\dot{I}_m$ is satisfied between the voltage phasor and the current phasor in the associated reference direction:

$$i=\sqrt{2}I\cos(\omega t+\varphi_i)=\sqrt{2}\,\mathrm{Re}[\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]$$
$$=C\frac{\mathrm{d}u}{\mathrm{d}t}=C\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{2}U\cos(\omega t+\varphi_i)=C\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{2}\,\mathrm{Re}[\dot{U}\mathrm{e}^{\mathrm{j}\omega t}]=\sqrt{2}\,\mathrm{Re}[\mathrm{j}\omega C\dot{U}\mathrm{e}^{\mathrm{j}\omega t}]\rightarrow\dot{I}=\mathrm{j}\omega C\dot{U} \tag{3.6.12}$$

$$\dot{U}=U\angle\varphi_u=\frac{1}{\mathrm{j}\omega C}\dot{I}=\frac{1}{\omega C}\angle-90^\circ I\angle\varphi_i=\frac{1}{\omega C}I\angle(\varphi_i-90^\circ) \tag{3.6.13}$$

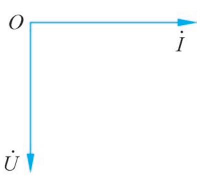


Figure 3.6.6 Vector diagram

The effective value or amplitude meets $U=\frac{1}{\omega C}$ or $U_m=\frac{1}{\omega C}I_m$, and the voltage phase lags behind the current phase by 90°, $\varphi_u=\varphi_i-90°$. As shown in Figure 3.6.6.

4. Impedance/admittance—phasor form of Ohm's law

The phasor form of impedance:

$$Z=\frac{\dot{U}}{\dot{I}}=\frac{U}{I}\angle(\varphi_u-\varphi_i)=|Z|\angle\varphi_z=R+\mathrm{j}X \tag{3.6.14}$$

The phasor form of admittance:

$$Y=\frac{\dot{I}}{\dot{U}}=\frac{I}{U}\angle(\varphi_i-\varphi_u)=|Y|\angle\varphi_y=G+\mathrm{j}B \tag{3.6.15}$$

The Phasor Form of Ohm's Law

$$\dot{U}=Z\dot{I}\quad\text{or}\quad\dot{U}_m=Z\dot{I}_m \tag{3.6.16}$$

$$\dot{I}=Y\dot{U}\quad\text{or}\quad\dot{I}_m=Y\dot{U}_m \tag{3.6.17}$$

(1) Impedance/admittance of resistor

$$\begin{cases} Z = \dfrac{\dot{U}}{\dot{I}} = R \\ Y = \dfrac{\dot{I}}{\dot{U}} = G \end{cases} \tag{3.6.18}$$

The impedance/admittance of a resistor has only the real part, that is, the resistance/conductance.

(2) Impedance/admittance of inductor

$$\begin{cases} Z = \dfrac{\dot{U}}{\dot{I}} = \mathrm{j}\omega L = \mathrm{j}X \rightarrow X = \omega L \\ Y = \dfrac{\dot{I}}{\dot{U}} = \dfrac{1}{\mathrm{j}\omega L} = -\mathrm{j}\dfrac{1}{\omega L} = \mathrm{j}B \end{cases} \tag{3.6.19}$$

then

$$B = -\frac{1}{\omega L}$$

The impedance/admittance of the inductor has only the imaginary part, i. e., the reactance/susceptance, generally known as the inductive reactance/inductance.

The inductance/inductance is not only related to the inductance L, but also to the angular frequency ω.

(3) Impedance/admittance of capacitor

$$Z = \frac{\dot{U}}{\dot{I}} = \frac{1}{\mathrm{j}\omega C} = -\mathrm{j}\frac{1}{\omega C} = \mathrm{j}X$$

then

$$\begin{cases} X = -\dfrac{1}{\omega C} \\ Y = \dfrac{\dot{I}}{\dot{U}} = \mathrm{j}\omega C = \mathrm{j}B \\ B = \omega C \end{cases} \tag{3.6.20}$$

The impedance/admittance of a capacitor consists only of the imaginary part, i. e., the reactance/susceptance, generally referred to as the capacitive reactance/conductance.

The capacitive reactance/susceptance is not only related to the capacitor C, but also to the angular frequency ω.

Example 3. 17 In the circuit shown in Figure 3. 6. 7, the reading of AC ammeter A_1 and A_2 are both known to be 10mA, so find the reading of AC ammeter A.

Solution: Assume parallel branch voltage $\dot{U}=U\angle 0°$;

$$\dot{I}_1=\frac{\dot{U}}{R}=\frac{U}{R}\angle 0°=10\angle 0°$$

$$\dot{I}_2=\mathrm{j}\omega C\dot{U}=\omega CU\angle 90°=10\angle 90°$$

$$\dot{I}=\dot{I}_1+\dot{I}_2=10\angle 0°+10\angle 90°=10+\mathrm{j}10=14.14\angle 45°$$

The AC ammeter A reads 14.14mA, as illustrated in Figure 3.6.8.

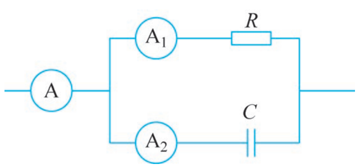


Figure 3.6.7 Example 3.17

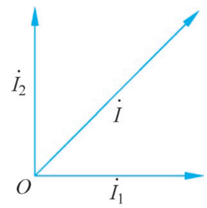


Figure 3.6.8 Illustration for Example 3.17

5. Phasor model of independent power supply

For an independent power source with the same angular frequency in a sinusoidal steady-state circuit, the VCR equation is expressed in terms of voltage or current phasor

(1) Independent voltage source.

VCR equation $\dot{U}=\dot{U}_S$ or $\dot{U}_m=\dot{U}_{Sm}$, thus:

$$u=\sqrt{2}U\cos(\omega t+\varphi_u)=\sqrt{2}\,\mathrm{Re}[\dot{U}\mathrm{e}^{\mathrm{j}\omega t}]$$

$$=u_S=\sqrt{2}U_S\cos(\omega t+\varphi_{uS})=\sqrt{2}\,\mathrm{Re}[\dot{U}_S\mathrm{e}^{\mathrm{j}\omega t}]$$

Therefore:

$$\dot{U}=\dot{U}_S \tag{3.6.21}$$

(2) Independent Current Source.

VCR Equation: $\dot{I}=\dot{I}_S$ or $\dot{I}_m=\dot{I}_{Sm}$, thus:

$$i=\sqrt{2}I\cos(\omega t+\varphi_i)=\sqrt{2}\,\mathrm{Re}[\dot{I}\mathrm{e}^{\mathrm{j}\omega t}]$$
$$=i_S=\sqrt{2}I_S\cos(\omega t+\varphi_{iS})=\sqrt{2}\,\mathrm{Re}[\dot{I}_S\mathrm{e}^{\mathrm{j}\omega t}] \tag{3.6.22}$$

Therefore:

$$\dot{I}=\dot{I}_S$$

6. Phasor model of controlled power supply

In a sinusoidal steady-state circuit with any controlled power supply, the voltage phasor and current phasor in the associated reference direction satisfy the following condition:

(1) VCVS.

VCR equation: $\dot{I}_1=0$, $\dot{U}_2=\mu\dot{U}_1$ or $\dot{I}_{m1}=0, \dot{U}_{m2}=\mu\dot{U}_{m1}$ it follows that

$$i_1=\sqrt{2}I_1\cos(\omega t+\varphi_{i1})=\sqrt{2}\,\mathrm{Re}[\dot{I}_1 e^{j\omega t}]=0 \tag{3.6.23a}$$

Then:

$$\begin{aligned}\dot{I}_1&=0\\ u_2&=\sqrt{2}U_2\cos(\omega t+\varphi_{u2})=\sqrt{2}\,\mathrm{Re}[\dot{U}_2 e^{j\omega t}]\\ &=\mu u_1=\mu\sqrt{2}U_1\cos(\omega t+\varphi_{u1})=\sqrt{2}\,\mathrm{Re}[\mu\dot{U}_1 e^{j\omega t}]\end{aligned} \tag{3.6.23b}$$

Then:

$$\dot{U}_2=\mu\dot{U}_1$$

(2) CCVS.

VCR equation: $\dot{U}_1=0, \dot{U}_2=r\dot{I}_1$ or $\dot{U}_{m1}=0, \dot{U}_{m2}=r\dot{I}_{m1}$, it follows that

$$u_1=\sqrt{2}U_1\cos(\omega t+\varphi_{u1})=\sqrt{2}\,\mathrm{Re}[\dot{U}_1 e^{j\omega t}]=0 \tag{3.6.24a}$$

Then:

$$\begin{aligned}\dot{U}_1&=0\\ u_2&=\sqrt{2}U_2\cos(\omega t+\varphi_{u2})=\sqrt{2}\,\mathrm{Re}[\dot{U}_2 e^{j\omega t}]\\ &=ri_1=r\sqrt{2}I_1\cos(\omega t+\varphi_{i1})=\sqrt{2}\,\mathrm{Re}[r\dot{I}_1 e^{j\omega t}]\end{aligned} \tag{3.6.24b}$$

Then:

$$\dot{U}_2=r\dot{I}_1$$

(3) VCCS.

VCR equation: $\dot{I}_1=0, \dot{I}_2=g\dot{U}_1$ or $\dot{U}_{m1}=0, \dot{U}_{m2}=r\dot{I}_{m1}$, it follows that

$$i_1=\sqrt{2}I_1\cos(\omega t+\varphi_{i1})=\sqrt{2}\,\mathrm{Re}[\dot{I}_1 e^{j\omega t}]=0 \tag{3.6.25a}$$

Then:

$$\begin{aligned}\dot{I}_1&=0\\ i_2&=\sqrt{2}I_2\cos(\omega t+\varphi_{i2})=\sqrt{2}\,\mathrm{Re}[\dot{I}_2 e^{j\omega t}]\\ &=gu_1=g\sqrt{2}U_1\cos(\omega t+\varphi_{u1})=\sqrt{2}\,\mathrm{Re}[g\dot{U}_1 e^{j\omega t}]\end{aligned} \tag{3.6.25b}$$

Then:

$$\dot{I}_2=g\dot{U}_1$$

(4) CCCS.

VCR equation: $\dot{U}_1=0, \dot{I}_2=\beta\dot{I}_1$ or $\dot{U}_{m1}=0, \dot{I}_{m2}=\beta\dot{I}_{m1}$, it follows that

$$u_1=\sqrt{2}U_1\cos(\omega t+\varphi_{u1})=\sqrt{2}\,\mathrm{Re}[\dot{U}_1 e^{j\omega t}]=0 \tag{3.6.26a}$$

Then:

$$\dot{U}_1 = 0$$

$$i_2 = \sqrt{2} I_2 \cos(\omega t + \varphi_{i2}) = \sqrt{2} \mathrm{Re}[\dot{I}_2 \mathrm{e}^{\mathrm{j}\omega t}]$$

$$= \beta i_1 = \beta\sqrt{2} I_1 \cos(\omega t + \varphi_{i1}) = \sqrt{2} \mathrm{Re}[\beta \dot{I}_1 \mathrm{e}^{\mathrm{j}\omega t}] \quad (3.6.26b)$$

Then:

$$\dot{I}_2 = \beta \dot{I}_1$$

7. The phasor form of Kirchhoff's law

1) Phasor form of KCL

The algebraic sum of all branch current phasors flowing into any node in a sinusoidal steady-state circuit is equal to zero.

KCL equation: $\sum_{k=1}^{n} \pm \dot{I}_k = 0$ or $\sum_{k=1}^{n} \pm \dot{I}_{mk} = 0$, then

$$\sum_{k=1}^{n} \pm i_k = \sum_{k=1}^{n} \pm \sqrt{2} I_k \cos(\omega t + \varphi_{ik}) = \sum_{k=1}^{n} \pm \sqrt{2} \mathrm{Re}[\dot{I}_k \mathrm{e}^{\mathrm{j}\omega t}]$$

$$= \sqrt{2} \mathrm{Re}\left[\sum_{k=1}^{n} \pm \dot{I}_k \mathrm{e}^{\mathrm{j}\omega t}\right] = 0 \rightarrow \sum_{k=1}^{n} \pm \dot{I}_k = 0$$

2) Phasor form of KVL

The algebraic sum of voltage phasors around any closed loop in a sinusoidal steady-state circuit is equal to zero.

KVL equation: $\sum_{k=1}^{n} \pm \dot{U}_k = 0$ or $\sum_{k=1}^{n} \pm \dot{U}_{mk} = 0$, then

$$\sum_{k=1}^{n} \pm u_k = \sum_{k=1}^{n} \pm \sqrt{2} U_k \cos(\omega t + \varphi_{uk}) = \sum_{k=1}^{n} \pm \sqrt{2} \mathrm{Re}[\dot{U}_k \mathrm{e}^{\mathrm{j}\omega t}]$$

$$= \sqrt{2} \mathrm{Re}\left[\sum_{k=1}^{n} \pm \dot{U}_k \mathrm{e}^{\mathrm{j}\omega t}\right] = 0$$

$$\begin{cases} \sum_{k=1}^{n} \pm \dot{U}_k = 0 \\ \sum_{k=1}^{n} \pm \dot{I}_k = \sum_{k=1}^{n} \pm I_k \angle \varphi_{ik} = 0 \\ \sum_{k=1}^{n} \pm \dot{U}_k = \sum_{k=1}^{n} \pm U_k \angle \varphi_{uk} = 0 \end{cases} \quad (3.6.27)$$

this shows:

(1) The current/voltage phasor satisfies KCL/KVL;

(2) The current/voltage RMS value or amplitude does not meet KCL/KVL.

Example 3.18 In sinusoidal steady-state RLC series circuits, $u_S = 10\sqrt{2}\cos(\omega t)$ (V), $u_L = 3\sqrt{2}\sin(\omega t)$ (V), $u_C = 15\sqrt{2}\cos(\omega t + 180°)$ (V), find u_R.

Solution:

$$u_S = 10\sqrt{2}\cos(\omega t) \leftrightarrow \dot{U}_S = 10\angle 0^\circ$$

$$u_L = 3\sqrt{2}\sin(\omega t) = 3\sqrt{2}\cos(\omega t - 90^\circ) \leftrightarrow \dot{U}_L = 3\angle -90^\circ$$

$$u_C = 15\sqrt{2}\sin(\omega t + 180^\circ) = 15\sqrt{2}\cos(\omega t + 90^\circ) \leftrightarrow \dot{U}_C = 15\angle 90^\circ$$

$$\dot{U}_R = \dot{U}_S - \dot{U}_L - \dot{U}_C = 10\angle 0^\circ - 3\angle -90^\circ - 15\angle 90^\circ$$

$$= 10 + j3 - j15 = 10 - j12 = 15.6\angle -50^\circ(\text{V})$$

$$u_R = 15.6\sqrt{2}\cos(\omega t - 50^\circ)(\text{V})$$

As illustrated in Figure 3.6.9.

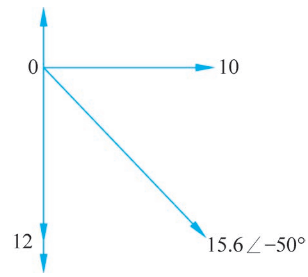


Figure 3.6.9 Illustration for Example 3.18

3.7 Phasor Analysis of Sinusoidal Steady-State Circuits

3.7.1 The Fundamental Method for Phasor Analysis of Sinusoidal Steady-State Circuits

(1) Phasor model of sinusoidal steady-state circuits involves the following:

① Unchanged circuit structure;

② Voltage and current are represented as voltage phasors and current phasors, with the reference direction remaining unchanged;

③ Component parameters change as follows: RLC parameters become impedance parameters, and voltage sources and current sources become voltage source phasors and current source phasors, with the reference direction remaining unchanged.

(2) According to the phasor model of the component and the phasor form of Kirchhoff's law, we can write the phasor equation and calculate the voltage phasor and current phasor.

(3) From the obtained voltage phasor and current phasor, the corresponding sinusoidal voltage and current can be determined.

Example 3.19 In the sinusoidal steady-state circuit shown in Figure 3.7.1, given $u_S = \sqrt{2}\cos(\omega t)(\text{V})$, find i when $\omega = 200\text{rad/s}$ and $\omega = 1000\text{rad/s}$, respectively.

Illustration for Example 3.19 is shown as Figure 3.7.2.

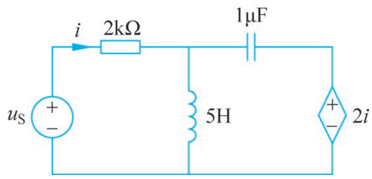


Figure 3.7.1 Example 3.19 circuit

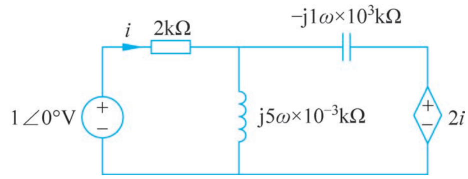


Figure 3.7.2 Illustration for Example 3.19

Solution: Based on the phasor model of sinusoidal steady-state circuits, formulate and solve the phasor equations:

$$2\dot{I} + j5\omega \times 10^{-3}(\dot{I} - \dot{I}_C) = 1\angle 0^\circ$$

$$-j1\omega \times 10^{3}\dot{I}_C + 2\dot{I} + j5\omega \times 10^{-3}(\dot{I}_C - \dot{I}) = 0$$

When $\omega = 200\text{rad/s}$,

$$2\dot{I} + j1(\dot{I} - \dot{I}_C) = 1\angle 0^\circ$$

$$-j5\dot{I}_C + 2\dot{I} + j1(\dot{I}_C - \dot{I}) = 0$$

$$(2 + j1)\dot{I} - j1\dot{I}_C = 1$$

$$(2 - j1)\dot{I} - j4\dot{I}_C = 0$$

$$\dot{I} = \frac{4}{6 + j5} = \frac{4\angle 0^\circ}{7.81\angle 39.8^\circ} = 0.51\angle -39.8^\circ(\text{mA})$$

When $\omega = 1000\text{rad/s}$,

$$2\dot{I} + j5(\dot{I} - \dot{I}_C) = 1\angle 0^\circ$$

$$-j1\dot{I}_C + 2\dot{I} + j5(\dot{I}_C - \dot{I}) = 0$$

$$(2 + j5)\dot{I} - j5\dot{I}_C = 1$$

$$(2 - j5)\dot{I} + j4\dot{I}_C = 0$$

$$\dot{I} = \frac{4}{18 - j5} = \frac{4\angle 0^\circ}{18.68\angle -15.5^\circ} = 0.21\angle 15.5^\circ(\text{mA})$$

Corresponding sinusoidal quantities:

When $\omega = 200\text{rad/s}$,

$$i = 0.51\sqrt{2}\cos(200t - 39.8^\circ)(\text{mA})$$

When $\omega = 1000\text{rad/s}$,

$$i = 0.21\sqrt{2}\cos(1000t + 15.5^\circ)(\text{mA})$$

3.7.2 Application of Superposition Theorem in Sinusoidal Steady-State Circuit Phasor Analysis

When the superposition theorem is applied to the analysis of sinusoidal steady-state circuits using phasors, only corresponding changes need to be made. The action of independent sources alone becomes the action of independent source phasors alone, the circuit transforms into a phasor model of the circuit (with impedance parameters varying with different angular frequencies), and the components of voltage/current transform into the phasor components of voltage/current.

Example 3.20 In the steady-state circuit as shown in Figure 3.7.3, given $u_{S1}=3\text{V}$, $u_{S2}=4\sqrt{2}\sin(2000t)(\text{V})$, find i.

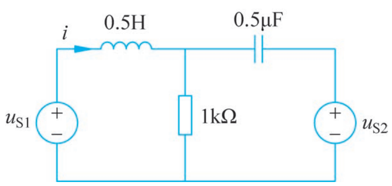


Figure 3.7.3 Circuit for Example 3.20

Solution:

When only $u_{S1}=3\text{V}$ acts independently, the phasor model (circuit) of the steady-state circuit is as shown in Figure 3.7.4.

Formulate the phasor equations (time-domain equations) and solve:

$$i_1=3/1=3(\text{mA})$$

When only $u_{S2}=4\sqrt{2}\sin(2000t)(\text{V})$ acts independently, the phasor model (circuit) of the steady-state circuit is as shown in Figure 3.7.5.

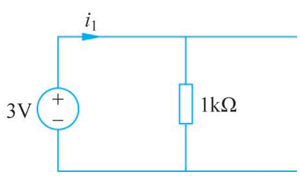


Figure 3.7.4 Illustration for Example 3.20(1)

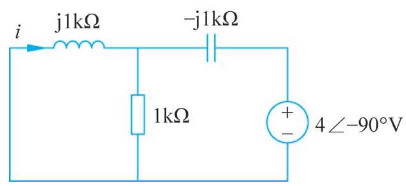


Figure 3.7.5 Illustration for Example 3.20(2)

Write the phasor equation and solve it:

$$\dot{I}_2=\frac{-1}{1+\text{j}1}\times\frac{4\angle-90^\circ}{-\text{j}1+\dfrac{\text{j}1}{1+\text{j}1}}=\frac{4\angle 90^\circ}{-\text{j}1(1+\text{j}1)+\text{j}1}=4\angle 90^\circ$$

Corresponding sinusoidal quantities:

$$i_2=4\sqrt{2}\cos(2000t+90^\circ)(\text{mA})$$

Superposition:

$$i=i_1+i_2=3+4\sqrt{2}\cos(2000t+90^\circ)(\text{mA})$$

3.7.3 Application of Thevenin/Norton Theorem in Phasor Analysis of Sinusoidal Steady-State Circuits

Similarly, in the phasor analysis of sinusoidal steady-state circuits, Thevenin's and Norton's theorems can be applied with corresponding transformations. The circuit is transformed into a phasor model of the circuit, the single-port network into a single-port phasor model, the open-circuit voltage/short-circuit current into the open-circuit voltage phasor/short-circuit current phasor, the equivalent resistance into the equivalent impedance, and Thevenin/Norton equivalent circuits into Thevenin/Norton equivalent phasor models.

Example 3.21 In the sinusoidal steady-state circuit shown in Figure 3.7.6, given $u_S=\sqrt{2}\cos(\omega t)(\mathrm{V})$, find i when $\omega=200\mathrm{rad/s}$ and $\omega=1000\mathrm{rad/s}$, respectively.

Solution:

The phasor model of the active single-port network, excluding the 2kΩ resistor branch, is shown in Figure 3.7.7.

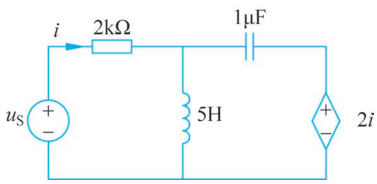


Figure 3.7.6 Circuit for Example 3.21

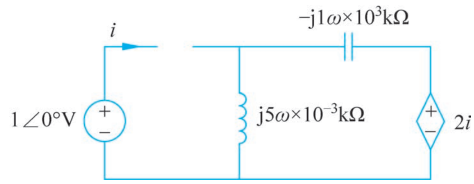


Figure 3.7.7 Illustration for Example 3.21 (1)

When finding the open circuit voltage phasor $\dot{U}_{oc}$c of N, $\dot{I}=0$, is shown in Figure 3.7.8.

$$\dot{U}_{oc}=1\angle 0°(\mathrm{V})$$

Use the external power supply method to find the equivalent impedance Z_0 of N→N_0, and add $\dot{U}$ to find $\dot{I}$, As shown in Figure 3.7.9.

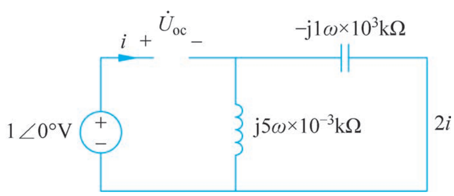


Figure 3.7.8 Illustration for Example 3.21(2)

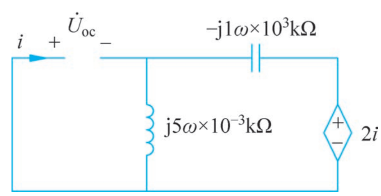


Figure 3.7.9 Illustration for Example 3.21(3)

$$\dot{I}=\frac{\dot{U}}{\mathrm{j}5\omega\times 10^{-3}}+\frac{\dot{U}-2\dot{I}}{-\mathrm{j}1\omega\times 10^{3}}$$

When $\omega=200\mathrm{rad/s}$, there are

$$\dot{I}=\frac{\dot{U}}{\mathrm{j}1}+\frac{\dot{U}-2\dot{I}}{-\mathrm{j}5}$$

$$Z_0=\frac{\dot{U}}{\dot{I}}=\frac{-2+\mathrm{j}5}{4}=-0.5+\mathrm{j}1.25(\mathrm{k\Omega})$$

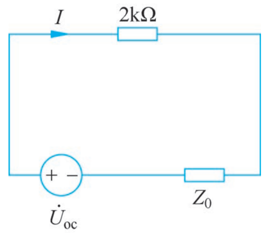


Figure 3.7.10　Illustration for Example 3.21(4)

When $\omega = 1000\text{rad/s}$, there are

$$\dot{I} = \frac{\dot{U}}{\mathrm{j}5} + \frac{\dot{U} - 2\dot{I}}{-\mathrm{j}1}$$

$$Z_0 = \frac{\dot{U}}{\dot{I}} = \frac{10 - \mathrm{j}5}{4} = 2.5 - \mathrm{j}1.25(\mathrm{k}\Omega)$$

The phasor model of the single-loop circuit is shown in Figure 3.7.10.

When $\omega = 200\text{rad/s}$, there is

$$\dot{I} = \frac{\dot{U}_{oc}}{2 + Z_0} = \frac{1\angle 0^\circ}{2 - 0.5 + \mathrm{j}1.25} = \frac{1\angle 0^\circ}{1.5 + \mathrm{j}1.25} = \frac{1\angle 0^\circ}{1.95\angle 39.8^\circ} = 0.51\angle -39.8^\circ(\mathrm{mA})$$

When $\omega = 1000\text{rad/s}$, there is

$$\dot{I} = \frac{\dot{U}_{oc}}{2 + Z_0} = \frac{1\angle 0^\circ}{2 + 2.5 - \mathrm{j}1.25} = \frac{1\angle 0^\circ}{4.5 - \mathrm{j}1.25} = \frac{1\angle 0^\circ}{4.67\angle -15.5^\circ} = 0.21\angle 15.5^\circ(\mathrm{mA})$$

The corresponding sine quantity:

When $\omega = 200\text{rad/s}$, there is

$$i = 0.51\sqrt{2}\cos(200t - 39.8^\circ)\mathrm{mA}$$

When $\omega = 1000\text{rad/s}$, there is

$$i = 0.21\sqrt{2}\cos(1000t + 15.5^\circ)\mathrm{mA}$$

3.7.4　Node Analysis in Sinusoidal Steady-State Circuit Phasor Analysis

In the phasor analysis of sinusoidal steady-state circuits using the node analysis method, only the corresponding variable transformations are required. The circuit is converted into a circuit phasor model, node voltages into node voltage phasors, self-conductance into self-admittance, mutual conductance into mutual admittance, sources into source phasors, and node equations into node phasor equations.

Example 3.22　In the sinusoidal steady-state circuit shown in Figure 3.7.11, given $u_S = \sqrt{2}\cos(\omega t)(\mathrm{V})$, find i when $\omega = 200\text{rad/s}$ and $\omega = 1000\text{rad/s}$, respectively.

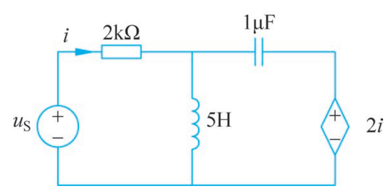


Figure 3.7.11　Circuit for Example 3.22

Solution:

Based on the phasor model of the sinusoidal steady-state circuit, with the reference node and node voltage phasors as shown in Figure 3.7.12, the control quantity for the controlled source phasor is transformed to $\dot{I} = (\dot{U}_1 - \dot{U}_2)/2$.

$$\dot{U}_1 = 1\angle 0^\circ(\mathrm{V}),\quad \dot{U}_3 = 2\dot{I} = \dot{U}_1 - \dot{U}_2$$

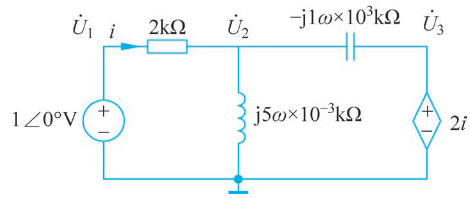


Figure 3.7.12 Illustration for Example 3.22

Node phasor equation for node 2:

$$-\frac{1}{2}\dot{U}_1+\left(\frac{1}{2}+\mathrm{j}(\omega\times 10^{-3}-\frac{1}{5\omega}\times 10^{3})\right)\dot{U}_2-\mathrm{j}\omega\times 10^{-3}\dot{U}_3=0$$

$$\dot{U}_2=\frac{\frac{1}{2}+\mathrm{j}\omega\times 10^{-3}}{\frac{1}{2}+\mathrm{j}\left(2\omega\times 10^{-3}-\frac{1}{5\omega}\times 10^{3}\right)}$$

When $\omega=200\mathrm{rad/s}$, there are

$$\dot{U}_2=\frac{0.5+\mathrm{j}0.2}{0.5+\mathrm{j}(0.4-1)}=\frac{0.5+\mathrm{j}0.2}{0.5-\mathrm{j}0.6}=\frac{0.539\angle 21.8^\circ}{0.781\angle -50.2^\circ}=0.69\angle 72^\circ(\mathrm{V})$$

$$\dot{I}=\frac{\dot{U}_1-\dot{U}_2}{2}=\frac{1\angle 0^\circ-0.69\angle 72^\circ}{2}=\frac{1-0.213-\mathrm{j}0.656}{2}$$

$$=0.394-\mathrm{j}0.328=0.51\angle -39.8^\circ(\mathrm{mA})$$

When $\omega=1000\mathrm{rad/s}$, there are

$$\dot{U}_2=\frac{0.5+\mathrm{j}1}{0.5+\mathrm{j}(2-0.2)}=\frac{0.5+\mathrm{j}1}{0.5+\mathrm{j}1.8}=\frac{1.118\angle 63.4^\circ}{1.868\angle 74.5^\circ}=0.599\angle -11.1^\circ(\mathrm{V})$$

$$\dot{I}=\frac{\dot{U}_1-\dot{U}_2}{2}=\frac{1\angle 0^\circ-0.599\angle -11.1^\circ}{2}=\frac{1-0.588-\mathrm{j}0.115}{2}$$

$$=0.206-\mathrm{j}0.058=0.21\angle -15.7^\circ(\mathrm{mA})$$

The corresponding sinusoidal quantities:

When $\omega=200\mathrm{rad/s}$, there is

$$i=0.51\sqrt{2}\cos(200t-39.8^\circ)(\mathrm{mA})$$

When $\omega=1000\mathrm{rad/s}$, there is

$$i=0.21\sqrt{2}\cos(1000t+15.7^\circ)(\mathrm{mA})$$

3.8 Frequency Characteristics of Sinusoidal Steady-State Circuits

3.8.1 Transfer Function and Frequency Characteristics of Sinusoidal Steady-State Circuits

The transfer function is the ratio of the output phase to the input phase of a sinusoidal steady-state circuit as a function of frequency. The frequency response characterizes the relationship between the amplitude and phase of a transfer function and frequency. The amplitude response represents the relationship between the amplitude of the transfer function and frequency, while the phase response denotes the relationship between the phase of the transfer function and frequency.

3.8.2 First-Order Low-Pass Characteristic

Example 3.23 Find the frequency characteristic of the first-order RC sinusoidal steady-state circuit shown in Figure 3.8.1.

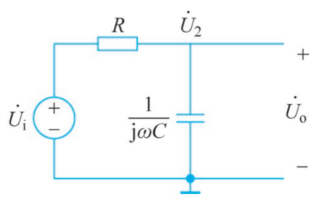


Figure 3.8.1 First-order low-pass circuit

Solution:

$$\dot{A}_u=\frac{\dot{U}_o}{\dot{U}_i}=\frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}=\frac{1}{1+j\omega RC}=\frac{1}{1+j2\pi fRC} \tag{3.8.1}$$

Assume that $A_u=1, f_0=\frac{1}{2\pi RC}$, there is

$$\dot{A}_u=\frac{A_u}{1+j\frac{f}{f_0}} \tag{3.8.2}$$

Amplitude-frequency characteristic:

$$|\dot{A}_u|=\frac{|A_u|}{\sqrt{1+\left(\frac{f}{f_0}\right)^2}} \tag{3.8.3}$$

Phase-frequency characteristic:

$$\varphi=0°-\arctan\left(\frac{f}{f_0}\right) \tag{3.8.4}$$

1. Qualitative analysis

When $f \ll f_0$, there is

$$|\dot{A}_u|\to|A_u|=1,\quad \varphi\to 0°$$

When $f=f_0$, there is

$$|\dot{A}_u|=\frac{|A_u|}{\sqrt{2}}=\frac{1}{\sqrt{2}},\quad \varphi=0°-\arctan 1=-45°$$

When $f\gg f_0$, there is

$$|\dot{A}_u|\to 0,\quad \varphi\to -90°$$

First-order low-pass characteristics (first-order hysteresis characteristics).

2. Bode plot analysis

In a coordinate system where the horizontal axis employs a logarithmic scale and the vertical axis uses a linear scale, the curves representing the amplitude-frequency response and phase-frequency response are known as the Bode plots.

$$20\lg|\dot{A}_u| = 20\lg|A_u| - 10\lg\left[1+\left(\frac{f}{f_0}\right)^2\right]$$
$$= \begin{cases} 20\lg1 - 10\lg1 = 0, f \ll f_0 \\ 20\lg1 - 10\lg2 = -3, f = f_0 \\ 20\lg1 - 20\lg\left(\frac{f}{f_0}\right) = -20\lg\left(\frac{f}{f_0}\right), \quad f \gg f_0 \end{cases} \tag{3.8.5}$$

The first-order low-pass amplitude-frequency Bode plot is shown in Figure 3.8.2.

$$\varphi = 0° - \arctan\left(\frac{f}{f_0}\right) = \begin{cases} 0°, & f \ll f_0 \\ 0° - \arctan1 = -45°, & f = f_0 \\ -90°, & f \gg f_0 \end{cases} \tag{3.8.6}$$

The first-order low-pass phase-frequency Bode plot is shown in Figure 3.8.3.

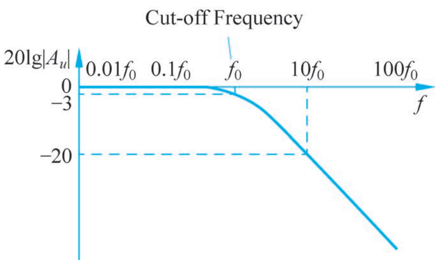


Figure 3.8.2 First-order low-pass gain Bode plot

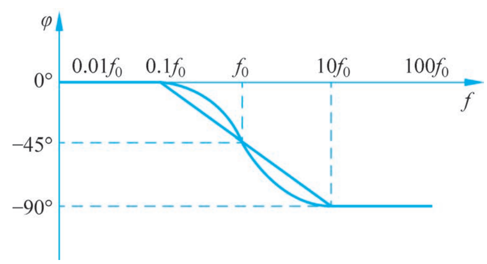


Figure 3.8.3 First-order low-pass phase-frequency Bode plot

3.8.3 First-Order High-Pass Characteristic

Example 3.24 Find the frequency characteristic of the first-order RC sinusoidal steady-state circuit shown in Figure 3.8.4.

Solution:

$$\dot{A}_u = \frac{\dot{U}_o}{\dot{U}_i} = \frac{R}{R+\frac{1}{j\omega C}} = \frac{1}{1+\frac{1}{j\omega RC}} = \frac{1}{1-j\frac{1}{2\pi fRC}} \tag{3.8.7}$$

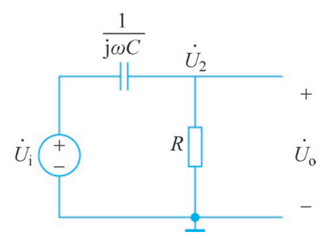


Figure 3.8.4 First-order high-pass circuit

Assume that $A_u = 1, f_0 = \frac{1}{2\pi RC}$, there is

$$\dot{A}_u = \frac{A_u}{1-j\frac{f_0}{f}} \tag{3.8.8}$$

Amplitude-frequency characteristic:

$$|\dot{A}_u| = \frac{|A_u|}{\sqrt{1+\left(\frac{f_0}{f}\right)^2}} \tag{3.8.9}$$

Phase-frequency characteristic:

$$\varphi = 0^\circ - \arctan\left(-\frac{f_0}{f}\right) \tag{3.8.10}$$

1. Qualitative analysis

When $f \ll f_0$, there is

$$|\dot{A}_u| \to 0, \quad \varphi \to 0^\circ$$

When $f = f_0$, there is

$$|\dot{A}_u| = \frac{|A_u|}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad \varphi = 0^\circ - \arctan(-1) = 45^\circ$$

When $f \gg f_0$, there is

$$|\dot{A}_u| \to |A_u| = 1, \quad \varphi \to 90^\circ$$

First-order high-pass characteristics (First-Order lead characteristics).

2. Bode plot analysis

$$20\lg|\dot{A}_u| = 20\lg|A_u| - 10\lg\left[1+\left(\frac{f_0}{f}\right)^2\right] = \begin{cases} 20\lg1 - 20\lg\left(\frac{f_0}{f}\right) = 20\lg\left(\frac{f}{f_0}\right), & f \ll f_0 \\ 20\lg1 - 10\lg2 = -3, & f = f_0 \\ 20\lg1 - 10\lg1 = 0, & f \gg f_0 \end{cases} \tag{3.8.11}$$

The first-order high-pass amplitude-frequency Bode plot is shown in Figure 3.8.5.

$$\varphi = 0^\circ - \arctan\left(-\frac{f_0}{f}\right) = \begin{cases} 90^\circ, & f \ll f_0 \\ 0^\circ - \arctan(-1) = 45^\circ, & f = f_0 \\ 0^\circ, & f \gg f_0 \end{cases} \tag{3.8.12}$$

The first-order low-pass phase Bode plot is illustrated in Figure 3.8.6.

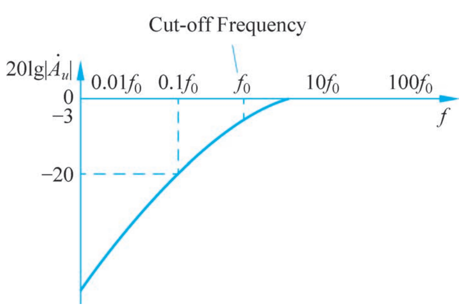


Figure 3.8.5 First-order high-pass amplitude-frequency Bode plot

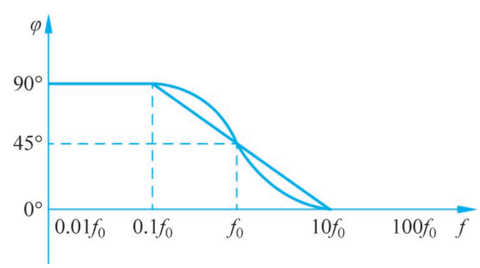


Figure 3.8.6 First-order low-pass phase Bode plot

3.9 Simulation: Thevenin Equivalent Circuits and Norton Equivalent Circuits

1. Experimental requirements and objectives

(1) Find the Thevenin equivalent circuit or Norton equivalent circuit of a linear active two-terminal network.

(2) Mastery of Thevenin's theorem and Norton's theorem.

2. Experimental principle

According to Thevenin's theorem and Norton's theorem, any linear two-terminal network with sources can be equivalently represented either as an actual voltage source in series with a resistor, consisting of an ideal voltage source, or as an actual current source in parallel with a resistor, comprising an ideal current source. The value of this ideal voltage source is equal to the open-circuit voltage at the ports of the two-terminal network, and the value of this ideal current source is equal to the short-circuit current at the ports of the two-terminal network. The value of this resistance is the equivalent resistance between the two ports after setting all the independent sources in the active network to zero. According to the law of interchangeability between two real power sources, this resistance is actually equal to the ratio of open-circuit voltage to short-circuit current.

3. Experimental circuit

The active two-terminal linear network is shown in Figure 3. 9. 1.

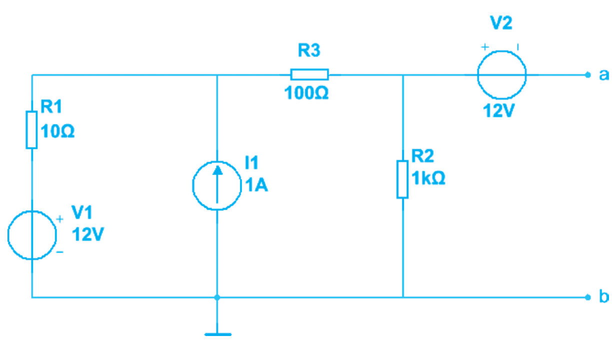


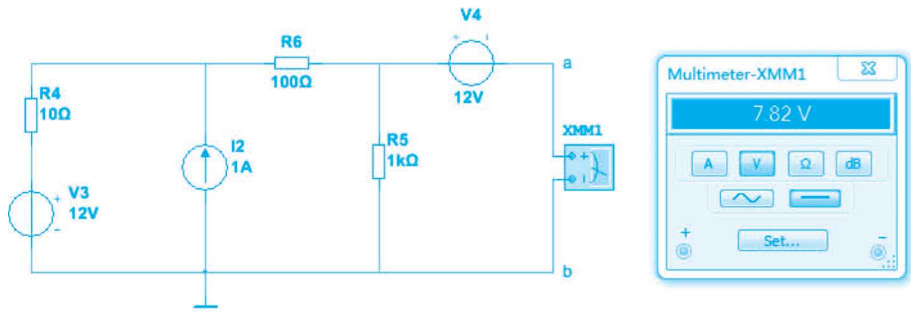
Figure 3. 9. 1 Active two-terminal linear network

4. Experimental step

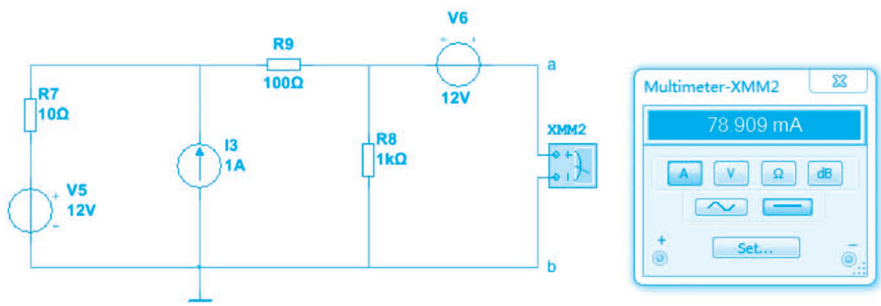
(1) Edit Figure 3. 9. 2 in the Circuit Window, where the nodes at points a and b are obtained through the initiation of the "Place Junction" command found in the "Place" menu; to get a, b text logo, start "Place Text" in the "Place" menu, and then enter the desired text in the determined position.

(2) Take out the multimeter from the instrument column and set it to the DC voltage block, connect it to the points a and b points and measure the open-circuit voltage. Measure the open-circuit voltage $U_{ab}=7.820\text{V}$, as shown in Figure 3.9.2(a).

(3) Set the multimeter to the DC current block, measure the short-circuit current I_s, measured short-circuit current $I_s=78.909\text{mA}$, as shown in Figure 3.9.2(b).



(a)



(b)

Figure 3.9.2 Circuit window editorial graph(1)

(4) Find the equivalent resistance of a two-terminal network.

Method 1: Through the measured open-circuit voltage and short-circuit current, the equivalent resistance of the two-terminal network can be obtained.

$$R_0=\frac{U_{ab}}{I_s}=\frac{7.820}{78.909}=0.0991\text{k}\Omega=99.1\Omega$$

Method 2: Replace all independent sources in the two-terminal network with zero, which means substituting voltage sources with a short circuit and current sources with an open circuit. Directly measure the resistance between points a and b using the ohmmeter function of the multimeter. The measured resistance $R_o=99.099\approx 99.1\Omega$, as shown in Figure 3.9.3.

(5) Draw the equivalent circuit. The Thevenin equivalent circuit is shown in Figure 3.9.4(a) and the Norton equivalent circuit is shown in Figure 3.9.4(b).

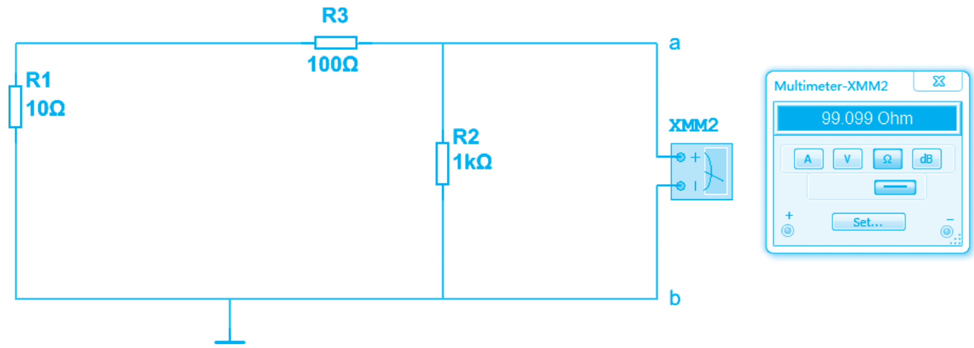
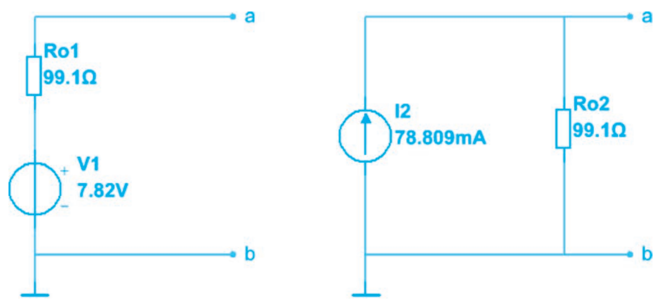


Figure 3.9.3 Circuit window editorial graph(2)



(a) (b)

Figure 3.9.4 Thevenin equivalent circuit and Norton equivalent circuit

Problems

3.1 The circuit is shown in Figure P3.1, find:

(a) How many linearly independent KVL equations can be written for the network in the figure?

(b) How many linearly independent KCL equations can be written for the network in the figure?

(c) Write a set of KVL and KCL equations for the network.

3.2 Using the Superposition Theorem, find the U_O of the circuit shown in Figure P3.2.

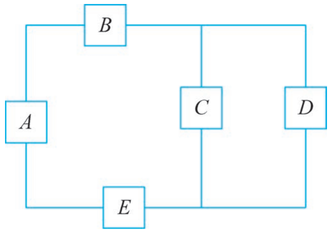


Figure P3.1

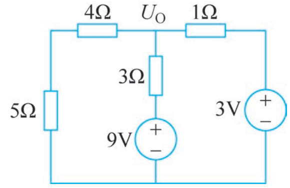


Figure P3.2

3.3 Using the Superposition Theorem, find the U_O of the circuit shown in Figure P3.3.

3.4 The two circuits in Figure P3.4 are equivalent, i.e., they have the same U and I relationship at the ports. Find the U_T, R_T.

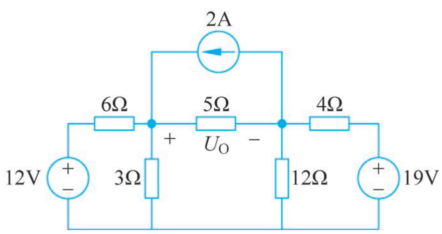


Figure P3.3

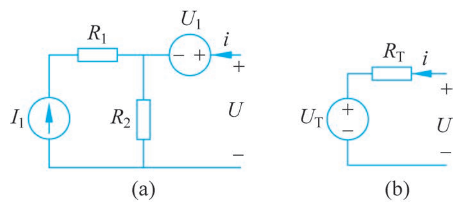


Figure P3.4

3.5 Find the Thevenin equivalent circuit of the circuit shown in Figure P3.5.

3.6 Determine the Thevenin equivalent circuit of the left-side network relative to the terminal pair aa'as shown in the circuit of Figure P3.6.

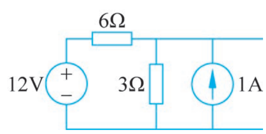


Figure P3.5

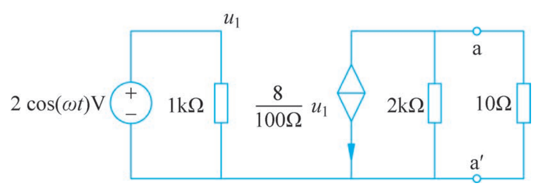


Figure P3.6

3.7 Find the Norton equivalent of the circuit shown in Figure P3.7.

3.8 Find the Norton equivalent of the circuit shown in Figure P3.8.

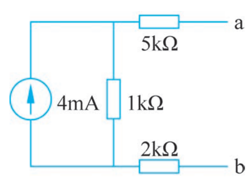


Figure P3.7

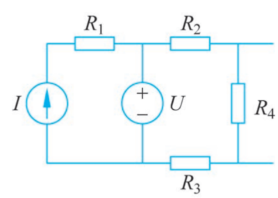


Figure P3.8

3.9 Determine the Norton equivalent circuit of the left-side network relative to the terminal pair aa'as illustrated in the circuit of Figure P3.9.

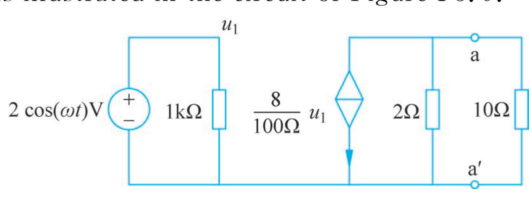


Figure P3.9

3.10 Find the time constant and cutoff frequency of the circuit shown in Figure P3.10. Where $R_S=1\text{k}\Omega$, $R_P=10\text{k}\Omega$, and $C_S=1\mu\text{F}$.

3.11 Find the time constant and cutoff frequency of the circuit shown in Figure P3.11. Where $R_S=1\text{k}\Omega$, $R_P=10\text{k}\Omega$, and $C_P=3\mu\text{F}$.

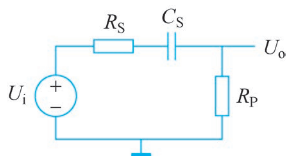


Figure P3.10

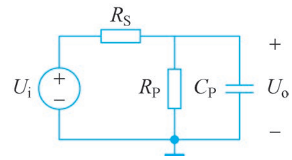


Figure P3.11

3.12 In the circuit shown in Figure P3.12, where $R_S=4.7\text{k}\Omega$, $R_P=25\text{k}\Omega$, and $C_P=120\text{pF}$, find the cutoff frequency f_H.

3.13 Find the cutoff frequency and bandwidth of the circuit shown in Figure P3.13. Where $R_S=1\text{k}\Omega$, $R_P=10\text{k}\Omega$, $C_S=1\mu\text{F}$, $C_P=3\text{pF}$.

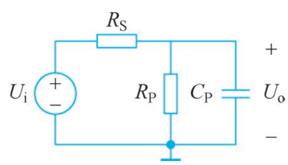


Figure P3.12

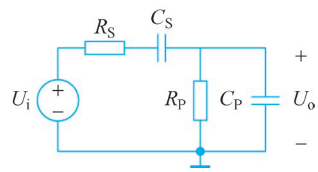


Figure P3.13

3.14 As the circuit shown in Figure P3.14, it is known that $U_1=40\text{V}$, $U_2=75\text{V}$, $R_1=20\text{k}\Omega$, $R_2=60\text{k}\Omega$, $R_3=8\text{k}\Omega$, $R_4=40\text{k}\Omega$, $R_5=160\text{k}\Omega$, $C=0.25\mu\text{F}$. The switch has been closed at position 1 for a long time, at $t=0$ the switch is turned to end 2, try to find the capacitor voltage $u_C(t)$ at $t\geqslant 0$.

3.15 The sinusoidal current $i_1(t)=20\cos(\omega t-30°)\text{A}$, $i_2(t)=40\cos(\omega t+60°)\text{A}$ and $i_3(t)=i_1(t)+i_2(t)$ is known, try to find the phase of $i_3(t)$.

3.16 In the circuit shown in Figure P3.16, it is known that $u_S=750\cos(5000t+30°)\text{V}$, $R=90\Omega$, $L=32\text{mH}$, $C=5\mu\text{F}$, try to find the steady state current i by using the phase method.

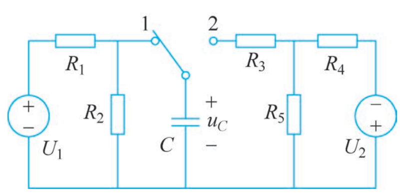


Figure P3.14

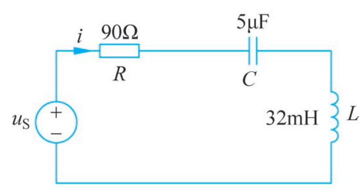


Figure P3.16