Chapter 1 Basic concepts

内容提要

- 图与简单图的基本概念。
- 图的关联矩阵、邻接矩阵。
- 图的简单变换，包括剖分、并、连图、线图、路树等概念。
- 图的谱的概念和几类简单图的谱的计算。
- 代数中关于矩阵的特征多项式、瑞利商等相关结论。

1.1 Graph and simple graph

Examples of graph are not difficult to find. For one, a road map can be interpreted as a graph, the vertex are the junctions and the edges are the stretch of road from one junctions to another, similarly an electrical circuit may give us a graph in which the vertex are the terminals and the edges are the wires. This graph is different from the lines and triangles, cycles in the geometry, and the painting either. Here, the graph we talk about is present a kind of relation on a set. For more the exact definition readers may read *Discrete Mathematics*. It is customary to represent a graph G by drawing on paper. A graph G is an ordered pair of disjoint sets $(V(G), E(G), \psi_G)$, here the set $V(G)$, $E(G)$ are the vertex and the edge set, ψ_G is the incident functions on $V(G)$ and $E(G)$, that is, if $\psi_G(e) = uv$ we say e *incident* with u and v. The vertices u and v are the end vertices of edge e, in other words, uv is an edge of G, we say u and v are *adjacent*. Two edges are *adjacent* if they have exactly one common end vertex.

We give an example to familiar the reader with the graph and associated terminologies.

$G = (V(G), E(G), \psi_G)$, here $V(G) = (v_1, v_2, v_3, v_4)$, $E(G) = (e_1, e_2, e_3, e_4, e_5)$ and ψ_G is defined as $\psi_G(e_1) = v_1v_2$, $\psi_G(e_2) = v_2v_3$, $\psi_G(e_3) = v_3v_4$, $\psi_G(e_4) = v_4v_1$, $\psi_G(e_5) = v_2v_4$, then this graph is showed in Fig 1.1.

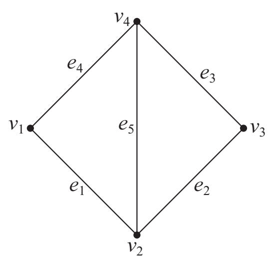

Fig 1.1 a simple graph

If more than one edge incident the same vertex, then we call graph has *multi-edges*, and if the end vertices are same of an edge, then we call the edge is a *loop*. In this book, we only think about the edges that do not have a direction. If a undirected graph without loops and multi-edges we call this graph is a *simple graph*. The number of vertices of a graph G we denoted as *the order* of G; the number of edges of a graph G we denoted as *the size* of G. For convince, we take n as the order of a graph and m the size of a graph in this book. Usually we denote $n = |V(G)|$ and $m = |E(G)|$.

Now, we denote several kind of graphs that has very interesting properties:

If a graph of order n without edge we call it *an empty graph* write as E^n.

A graph of order n and size C_n^2 or $C(n, 2)$ in some books is called a complete graph. This graph is denoted by K^n. In K^n, every two vertices are adjacent, the graph $K^1 = E^1$ is said to be *trivial graph*. A graph G is called a bipartite graph with vertex class V_1, V_2, if, and each edge joins a vertex of V_1 to V_2. K_{mn} is a complete bipartite graph on $n + m$ vertices, in fact, it is a special case of general bipartite graph. The set of vertices adjacent to a vertex $u \in G$ is denoted by $\Gamma(u)$. The *degree* of a vertex u is denoted as $d(u) = |\Gamma(u)|$. *The minimum degree* of a graph is denoted by $\delta(G)$ and δ for short; *the maximum degree* by $\Delta(G)$ and Δ for short, if $\Delta = \delta = k$ we call this graph is k regular.

Example

In fig 1.1 minimal degree $\delta = 2$, maximal degree $\Delta = 3$.

We say that $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In this case, we write $H \subseteq G$. If H contains all edges of G that join two vertices in V' then H is said to be the subgraph induced by V' and is denoted by $G[V']$. If H contains all the vertices that incident with the edges E' then we say H is a sub-graph induced by E' and is denoted by $G[E']$. If $V' = V$ then H is said to be *a spanning subgraph* of G. To example, we give several subgraphs.

Example

A subgraph, an induced subgraph by edges, an induced subgraph by vertex and a spanning subgraph. $V_1 = \{v_1, v_2, v_4\}, E_1 = \{e_1, e_3, e_5\}$ in graph 1.1 $H, G[V1], G[E1]$ are Fig 1.2, Fig 1.3 and Fig 1.4 respectively.

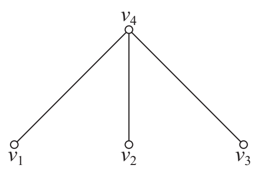

Fig 1.2 subgraph of G

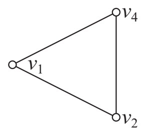

Fig 1.3 vertex induced graph $G[V1]$

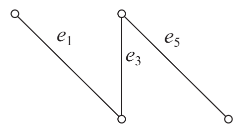

Fig 1.4 edge induced graph $G[E1]$

In order to give readers a wide bases, we give more terminologies, we call a subgraph A is *a clique* if $A \subseteq V(G)$ and every pairs of vertices are adjacent. In the sequent sections, we will know it is a *complete graph*. On the other hand, if non vertices are adjacent in A we call A is *an independent set*. We denote $c(G)$ is the clique number of a graph witch is the maximal number of vertices of all cliques of G, and $\alpha(G)$ is the maximal independent number of a graph.

Similar to the clique and independent set on vertex, we can extent these definitions to the edge set, we call *complete matching* and *an edge covering*. We will study these in chapter 6 for more information.

1.2 Graph operations

Sometimes, we study the properties of a graph by studying another graph get by transforming the original graph. We will study the spectrum of graphs in chapter 1. calculate the number of its *spanning tree* of a graph in chapter 3, calculate the matching number of graphs, study the relation between *the matching polynomial* of a graph and *the characteristic polynomial* of its *path tree* and study the coloring number of a graph in chapter 6. Here, we first present some operations on graphs.

1. deleting an edge or a vertex from G, denoted as G-e or G-v.
2. subdivision an edge or split a vertex.
3. put two graphs together, write as $G_1 \cup G_2$.
4. contracting graph by an edge, delete an edge and put two end vertex together all other vertex and edges keep same.
5. complete product(some books call it joint)$G_1 \bigtriangledown G_2$ of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

In the following chapter, we may study the different polynomials defined on these transformations. In this section, we study the properties on following transformations.

Definition 1.1 *The complement of G, denoted by G^c, is the graph with $V(G) = V(G^c)$ such that two vertices are adjacent in G^c if and only if their are not adjacent in G.*

Obviously, $|V(G)| = |V(G^c)|$ and $|E(G)| + |E(G^c)| = C(n, 2) = n(n-1)/2$. We will have more interesting results in the later chapters about the complement graph and with itself.

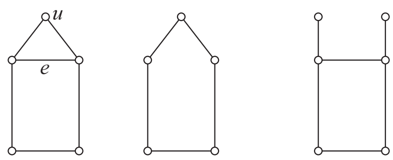

Fig 1.5 graph G and the graph delete e and split from u

Example

We give another example for subdivision and contract by an edge of graph.

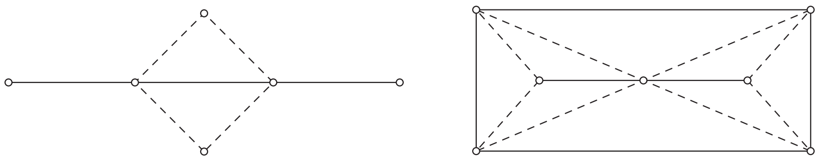

Fig 1.6 a simple graph and its line graph

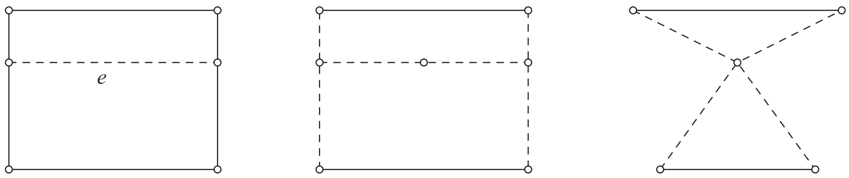

Fig 1.7 the graph obtained by subdivision e and split from e

Example

We construct a new graph from the original one by a simple transformations. Besides these, we also have several special operations these are very important in studying the graph properties. The *line graph L(G)* of an undirected graph G is another graph $L(G)$ that represents the adjacency between edges of G. The line graph is also sometimes called the edge graph, the adjoint graph, the interchange graph, or the derived graph of G. In *Spectra of Graphs*, readers may find more graph transformations like *the direct sun*, *the complete product*, *the product* and *the total graph*, etc.

Definition 1.2 *Given a graph G, its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G; and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint (“are adjacent”) in G.*

We give an example for general simple graph G and its line graph $L(G)$. We can easily find an edge of G correspond to an vertex of its line graph $L(G)$. In the graph above edge e correspond to the vertex v of $L(G)$. The degree of v of $L(G)$ satisfies below formula

$$d(v) = d(v_i) + d(v_j) - 2, e = (uv)$$

Obviously, the edge set of $L(G)$ is the edge set of G. The size of G became the order of its line graph. The size of $L(G)$ satisfy following equation.

$$E(L(G)) = \sum_{i=1}^{m} d(v_i)$$

We may give this formula and let the readers proof this as an exercise in the end of this chapter.

$$E(L(G)) = \frac{1}{2}\sum_{i=1}^{m} d(e_i) = \frac{1}{2}(\sum_{i=1}^{n} d_i^2 - 2m)$$

In Fig 1.8, the number of edges of the line graph is 14. If G is *k-regular*, then $L(G)$ is $2k-2$ regular. Besides this, the maximal matching, the independent vertex set, the color number, the connectivity and the character polynomial of G and $L(G)$ are studied by many mathematicians. In section 1.6, we will prove that the eigenvalues of a line graph $L(G)$ are not less than -2. Here, we cite several results about the characteristic polynomials of regular graphs. In the next section, we will give the proof of this theorem.

Theorem 1.1([36]) *If G is a k-regular graph with n vertices and m edges, $P(G,\lambda)$ is the characteristic polynomial of its adjacent matrix, then*

$$\rho(L(G),\lambda) = (\lambda+2)^{m-n}\rho(G,\lambda-k+2)$$

It is interesting that the number of triangles in graph G and its line graph $L(G)$ has below relationship. Let us denote the triangle number of G and $L(G)$ as $\mathit{\Delta}(G)$ and $\mathit{\Delta}(L(G))$, respectively, then

$$\mathit{\Delta}(L(G)) = \mathit{\Delta}(G) + \sum_{i=1}^{n} C(d_i, 3)$$

where d_i is the degree of vertex v_i in G.

We give an example of this formula here.

A semi-regular bipartite graph is a bipartite graph, Let V_1, V_2 be two parts of $V(G)$, $d(v) = s$ if $v \in V_1$; $d(v) = t$ if $v \in V_2$, then Shu jinlong has following theorem:

Theorem 1.2([37]) *$L(G)$ is a connected regular graph if and only if G is a connected graph or semi-regular graph.*

Let $G_1 \bigtriangledown G_2$ (the complete product) denote the joint of G_1 and G_2 obtained by adding all possible edges uv,$u \in G_1$ and $v \in G_2$.

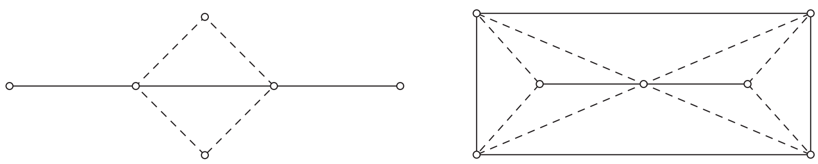

Fig 1.8 The number of triangles in G and its line graph

Theorem 1.3([38]) *Let G_1 and G_2 are k_1-regular graph and k_2-regular graph, respectively, and $k_1 - k_2 = n_1 - n_2$, where n_1 and n_2 are the order of G_1 and G_2, respectively,* *then the quai-Laplacian polynomial of* $G_1 \vee G_2$ *is* $L((G_1 \vee G_2), x) = \frac{(x-n_1-n_2-k_1-k_2)(n_2-n_1-x+2k_1)(n_1-n_2-x+2k_2)}{(x-k_1-k_2-2)(n_2-x+2k_2)(n_1-x+2k_2)} L(G_1, x-n_2)L(G_2, x-n_1).$

The matching polynomial and characteristic polynomial is connected by the graph and its *path tree.*(see chapter 6) Here, we only give the definition and several simple results.

Definition 1.3 *The path tree* $T(G,u)$ *of* G *take vertex* u *as its root, if:*

1. $V(T(G,u))$*=all the paths start from* u *include* u *itself;*

2. $E(T(G,u))$*=*(P_i, P_j) *if one path is contained in the other maximally.*

We give an example in Fig 1.9. Obviously, a path tree of P_n is P_n, Zhang hailiang in [39] gave the path tree of several type of graph and studied the relation of *the largest zero of matching polynomial.* In [4] and in [21], Ma haicheng proved that the largest zero of a graph's matching polynomial equals the largest zero of characteristic polynomial of its path tree. Zhang hailiang gave following properties of several path tree of certain graphs.

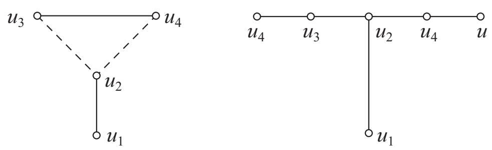

Fig 1.9 A simple graph and its path tree

Theorem 1.4 *1. The path tree of* C_n *is* P_{2n-1}*;*

2. The path tree of $Q(s,t)$ *is* $T_{s-1,s-1,t-1}$*, or* $T_{s-2,t-1,s+t-1}$ *or* $T_{i,t-1,s-1,t-1,j}$*, where* $i+j = s-1$*.*

Let the n vertices of the given graph G be $v_0, v_1, \ldots, v_n$. The Mycielski graph of G contains G itself as an isomorphic subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and another vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge v_iv_j of G, the Mycielski graph includes two edges, u_iv_j and v_iu_j.

Thus, if G has n vertices and m edges, $My(G)$ has $2n+1$ vertices and $3m+n$ edges. Mycielski's construction is applied to a 5-vertex cycle we get a graph which is called the Grotzsch graph. this graph has 11 vertices and 20 edges. The Grotzsch graph is the smallest triangle-free 4-chromatic graph (Chv á tal 1974). Zhang hai liang in [18] studied the matching polynomial and matching equivalent graphs of this graph.

1.3 Isomorphism

Two graph are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus $G = (V, E)$ is isomorphic to $G' = (V', E')$, we denoted by $G \cong G'$, or simply $G = G'$. If there is a bijection $\theta : V \to V'$ and $\phi : E(G) \to E(G')$ such that $\psi_G(e) = uv$ if and only if $\psi_{G'}(\phi(e)) = \theta(u)\theta(v)$, clearly isomorphic graphs has the same order and size, usually we do not distinguish between isomorphic graphs, unless we consider graphs with a distinguished or labeled set of vertices.

Definition 1.4 *A graph is said to be self-complementary if $G \cong G^c$.*

We have below properties about self-complement graphs.

Theorem 1.5 *A graph is self-complementary then $v \equiv 0, 1 mod(4)$.*

Fig 1.10 gives two isomorphic graph.

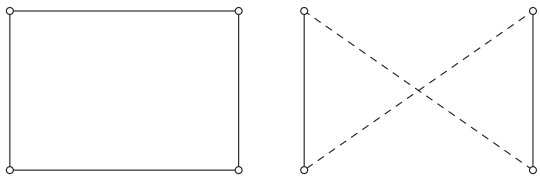

Fig 1.10 two isomorphic graph

1.4 Incident and adjacent matrix

A graph can be represent as a matrix in the computer science. This section we will give matrix theory used in graph theory and build a strong connection between matrix and a graph, first, we start this section with define the adjacency matrix of a graph:

Definition 1.5 *The adjacency matrix $A(G)$ of a simple graph G whose vertex set is $\{v_1, v_2, \ldots, v_n\}$is a square matrix of order n .Whose entry a_{ij} at the place (i, j) is equal to the numbers of edges incident with the v_i, v_j, for simple graph that is 0 or 1. We shall write $A = (a_{ij})$.*

Since this matrix is a symmetric matrix, then it has several properties as below:

Theorem 1.6 *All eigenvalues of A are real numbers.*

Proof. Let λ be an eigenvalue of A and P is the associated eigenvectors of λ. $\bar{\lambda}$ and $\bar{p}$ be the conjugate of λ and p, respectively, then

$$\lambda \bar{p}^t.p = \bar{p}^t(\lambda p) = \bar{p}^t Ap$$

since A is symmetric then

$$(\bar{A}\bar{p})^t p = (\bar{\lambda}\bar{p})^t p = \bar{\lambda}\bar{p}^t p$$

$$\lambda\bar{p}^t p = \bar{\lambda}\bar{p}^t p$$

and $\bar{p}^t p \geqslant 0$ so λ is real number.

We can also use the associate law of matrix multiplication and the equation

$$\bar{p}^t Ap = \lambda\bar{p}^t p$$

to proof this theorem.

Theorem 1.7 *For every symmetric matrix A there is an orthogonal matrix P such that* $P^t AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$*,* λ_i *are the eigenvalues of A.*

Proof. According to theorem 1.5, we know that A has an eigenvector v_1, we can assume $\|v_1\| = 1$, and by using the Gram-Schmidt procedure we can find an orthogonal basis $\mathcal{B}$=$\{v_1, v_2, \ldots, v_n\}$ with the eigenvector v_1 as the first element. Let $P_1 = \{v_2, v_3, \ldots, v_n\}$ the $dim(P_1) = n - 1$ Since v_1 is an eigenvector of T_A with the eigenvalue λ_1, then AP_1 also a symmetric transformation, by the introduction, $(v_1, v_2, \ldots, v_n)$ is a orthogonal basis for A. Then by the well know theorem of diagonalizable theorem we finish proving our proof.

The diagonalizable theorem is that if a matrix of order n has n different eigenvectors then this matrix can be diagonalizable.

Definition 1.6 *The incident matrix* $M(G)$ *of graph G is a* $n \times m$ *matrix* $M = M(G)$*, its row is the set of vertices and the columns is the set of edges, and whose entries are given by*

$$m_{ij} = \begin{cases} 1, & if\ v_i\ and\ e_j\ are\ incident \\ 0, & otherwise \end{cases}$$

Example

The adjacency matrix and the incident matrix of graph 1.1 are

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \ and \ \ M(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

respectively.

Definition 1.7 *A matrix is said to be totally unimodular if every minors of order k is* $0, -1, 1$.

In fact, we can easily proofed the incident matrix of a simple graph is totally unimodular by the induction on the order of the minors of the matrix.

Theorem 1.8 (Egervary 1931) *G is bipartite if and only if M is totally unimodular.*

The characteristic polynomial of adjacent matrix of a graph G is defined as the characteristic polynomial of G, write as $\rho(G,\lambda)$, sometimes $\rho(G)$ for short.

Definition 1.8 *The spectrum of a graph G is the set of numbers which are eigenvalues of $A(G)$, together with their multiplicities.If the distinct eigenvalues of $A(G)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{n-1}$,and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{n-1})$,then we shall write:*

$$\mathrm{Spec}G = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{n-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{n-1}) \end{pmatrix}$$

The spectrum of graph 1.1 is

$$\mathrm{Spect}(G) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Suppose that λ is an eigenvalue of A, then since A is real and symmetric, it follows that λ is real,and the multiplicity of λ as a root of the equation $\det(\lambda I - A) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ. The main question arising is this:how much information concerning the structure of G is contained in its spectrum, and how can this information be retrieved from the spectrum?

Theorem 1.9 (Hand-shaking lemma) *For a graph $\sum_{i=1}^{n} d(v_i) = 2\varepsilon$, where ε is size of a graph.*

Proof. Since every edge gives two degrees to a pair of adjacent vertices of a graph, so the sum of degree is twice of the numbers of E of G.

Corollary 1.1 (Hand-shaking theorem) *In any graph the number of odd degree vertices is even.*

Proof. Assume V_1, V_2 represent the odd degree vertices set and the even degree vertices set, respectively, by the Theorem 1.9 we have:

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2\varepsilon$$

The right side of this equation is even, as to the left side $\sum_{v \in V_2} d(v)$ is even , so $\sum_{v \in V_1} d(v)$ must be a even number, but in which every degree of vertex is odd, so in order to grantee the summation is even, the number of vertices must be a even number.

If $V(G) = (v_1, v_2, \ldots, v_n)$, then we say $d(v_1), d(v_2), \ldots, d(v_n)$ is the degree sequence of G. This sequence must have below property.

Theorem 1.10 *For positive integer sequence $d(v_1), d(v_2), \ldots, d(v_n)$ is a degree sequence of a graph if and only if $\sum_{i=1}^{n} d(v_i)$ is even.*

Proof. Necessity is obvious by the theorem 1.9. Now we prove the sufficient condition, for $\sum_{i=1}^{n} d(v_i)$ is even by the hand shaking theorem there must have even number of vertex which has odd degree, then we can construct a graph as below: For the even degree vertex v_i we draw $d(v_i)/2$ loops on v_i; for the odd degree vertices v_j we draw $(d(v_j)-1)/2$ loops and connect every two odd degree vertices with an edge, by the hand shaking theorem there are even number of odd degree vertices, hence, this graph satisfy the condition.

1.5 The spectrum of graph

In this section, we give an expression of characteristic polynomial. We explain the connection of graph structure and the coefficients of characteristic polynomial.Some of results come from the matrix theory directly.

Lemma 1.1([7]) *Let $A = (a_{ij}) \in R^{n\times n}$, then*

$$|\lambda I - A| = \lambda^n + \sum_{k=1}^{k} (-1)^k b_k \lambda^{n-k}$$

where $b_k (k = 1, 2, \ldots, n)$ is the summation over all principle minors of order k, especially,

$$b_1 = a_{11} + a_{22} + \ldots + a_{nn}, b_n = |A|$$

Proof. Let $E = (e_1, e_2, \ldots, e_n), A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$,where e_i and α_i are the $i-th$ columns of unity matrix E and matrix A, respectively, then

$$|\lambda I - A| = |(\lambda e_1 - \alpha_1, \lambda e_2 - \alpha_2, \ldots, \lambda e_n - \alpha_n)|$$

expand this determinant we have:

$$|\lambda I - A| = \lambda^n |(e_1, e_2, \ldots, e_n)| - \lambda_{n-1} \sum |e_1, \ldots, e_{i-1}, \alpha_i, e_{i+1}, \ldots, e_n| + \ldots$$
$$+(-1)^k \lambda^{n-k} \sum_{1\leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} |(\ldots, a_{i_1}, \ldots, a_{i_k}, \ldots)| + (-1)^n |A|$$

Where

$$|(\ldots, a_{i_1}, \ldots, a_{i_k}, \ldots)|$$

represent the two column of adjacent matrix of A, the others are columns of unity matrix I.

Theorem 1.11([4]) *Let* $\rho(G,\lambda) = |\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ *be the characteristic polynomial of an arbitrary undirected multi-graph* G*. then*

$$a_i^i = \sum_{U \in U_i} (-1)^{p(U)} \cdot 2^{c(U)} \quad (i = 1, 2, \ldots, n)$$

We call following graphs "elementary figure"

1. the graph K_2*,or*

2. every graph $C_q(q \geqslant 1)$*(loops being included with* $q = 1$*)*

call a "basic figure" U *every graph all of whose components are elementary figures; let* $p(U), c(U)$ *be the number of components and the number of circuits contained in* U*, respectively, and* U_i *denote the set of all basic figures contained in* G *having exactly* i *vertices.*

This theorem may be given the following form:

Define the "contribution" b *of an elementary figure* E *by* $b(K_2) = -1, b(C_q) = (-1)^{q+1} \cdot 2$ *and basic figure* U *by* $b(U) = \prod_{E \in U} b(E)$*,then* $(-1)^i a_i = \sum_{U \in U_i} b(U)$*.*

Proof. Let us first consider the absolute term

$$a_n = P_G(0) = (-1)^n |A| = (-1)^n |a_{ik}|$$

According to *Leibniz* definition of the determinants,

$$a_n = \sum_P (-1)^{n+I(P)} a_{1i_1} a_{2i_2} \ldots a_{ni_n}$$

For the sake of simplicity, let us first assume that there are no multiple arcs so that $a_{ik} = 0$ or 1 for all i, k. A term

$$S_P = (-1)^{n+I(P)} a_{1i_1} a_{2i_2} \ldots a_{ni_n}$$

of the sum is different from zero if and only if all of the arcs $(1, i_1), (2, i_2), \ldots, (n, i_n)$ are contained in G, P may be represented as a product:

$$P = (1i_1)(\ldots)(\ldots)(\ldots)$$

of disjoint cycles. Evidently, if $S_P \neq 0$, then to each of cycle of P there are corresponds a cycle in G: thus to P, there corresponds a direct sum of (non-intersecting) cycles containing all vertices of G,i.e., a linear directed sub-graph $L \in L_n$. Conversely: to each linear directed subgraph $L \in L_n$ there corresponds a permutation P and a term $S_P = \pm 1$, the sign depending only on the $e(L)$ of even cycles among all cycles of L:

$$S_P = (-1)^{n+e(L)}$$

obviously,

$$n + e(L) \equiv P(L)(mod2)$$

hence

$$a_n = \sum_P S_P = \sum_{L \in L_n} (-1)^{P(L)}$$

Now, the theorem remains valid even if $a_{ik} > 1$ is allowed:
consider the set of all distinct linear directed subgraph $L \in L_n$ connecting the n vertices of G in exactly the way prescribed by the cycle of a fixed permutation $P = (1i_1)(\ldots)(\ldots)(\ldots)$, it is clear that this set can be obtained by arbitrarily choosing for each k arcs from vertex k to vertex i_k, and doing so in every possible manner; and since for fixed k there are exactly a_{ki_k} possible choices, the total number of subgraph so obtained equals $a_{1i_1}a_{2i_2}\ldots a_{ni_n}$. thus the total contribution of all of these subgraphs to the sum $\sum\limits_{L \in L_n}(-1)^{P(L)}$ equals to $(-1)^{n+I(P)}a_{1i_1}a_{2i_2}\ldots a_{ni_n}$ summation with respect to all permutations P confirms the validity of in general case.

In order to complete the proof of suppose $1 \leqslant i \leqslant n$, ($i$ fixed). It is well know that $(-1)^i a_i$ equals to the sum of all principal minors of order i of A. Note that there is a $(1-1)$ correspondence between the set of these minors and the set of induced subgraph of G exactly having i vertices. By applying the result obtained above to each of the $\binom{n}{i}$ minors, and summing, the valid of theorem is established.

A spanning element graph of G is an elementary sub-graph which contains all vertices of G then

$$\det(A) = \sum(-1)^{p(U)}2^{c(U)}$$

where the summation over all spanning sub-graphs U of G. This theorem is proofed by Harary in 1962. Here gives its proof:
consider a term

$$\mathrm{sgn}(\pi)a_{1q_1}a_{2q_2},\ldots,a_{nq_n}$$

in the expansion of Det(A). The term vanishes, if for some $i \in \{1,2,\ldots,n\}, a_{iq_j} = 0$; that is if (v_i, v_{q_j}) is not an edge of G. In particular, the term vanish if π fixes any symbol. Thus the term corresponding to a permutation π is non-zero, then π can be expressed uniquely as the composition of disjoint cycles of length at least two;Each cycle (ij) of length two corresponds to the factors $a_{ij}a_{ji}$, and signifies a single edge $\{v_i, v_j\}$ in G. Each cycle $(pqr\ldots t)$ of length greater than two corresponds to the factors $a_{pq}a_{qr}\ldots a_{tp}$, and signifies a cycle $\{v_p v_q \ldots v_t\}$ in G, consequently each nonvanish term in the determinate expansion gives rise to an elementary sub-graph U of G,with $|V(U)| = |V(G)|$. The sign of a permutation π is $(-1)^{N_e}$, where N_e is the number of even cycles in π. If there are c_l cycles of length l, then the equation $\sum lc_l = n$ shows that the number N_o is the number of odd cycles is congruent to n module 2. Hence,

$$p(U) = n - (N_o + N_e) \equiv N_e(\text{mod}2)$$

so the sign of π is equal to $(-1)^{p(U)}$.

Theorem 1.12 *(Biggs Algebraic graph P.53) Suppose the bipartite graph G has an eigenvalue λ with multiplicity $m(\lambda)$, then $-\lambda$ is also an eigenvalue of G with same multiplicity.*

Proof. By the theorem 1.11, $(-1)^i a_i = \sum_U (-1)^{p(U)} 2^{c(U)}$. If G is bipartite then G has no odd cycle, and consequently, no elementary subgraph with an odd number vertices. It follows that the characteristic polynomial of G has the form

$$\rho(G, \lambda) = z^n + a_2 z^{n-2} + \ldots = z^\delta p(z^2)$$

where $\delta = 0$ or 1, and P is a polynomial function of z. Thus the eigenvalues which are zeros of ρ, have the required property.

Theorem 1.13 *For a simple connected graph G, λ_{max} be the largest eigenvalue of its characteristic polynomial(spectral radius), then G has no odd cycles if and only if $-\lambda_{max}$ also is an eigenvalue of $A(G)$, where $A(G)$ is the adjacent matrix of graph G.*

Readers may find the proof of this threom on page 83 *Spectra of Graphs Theory and Application* by Dragoš M.Cvetković. etc. Since if G is a bipartite graph if and only if G does not contain a odd cycle, above theorem gives a very interesting relation between the spectrum of graphs and its structure.

Corollary 1.2 (Coulson and Rushbrooke 1940) *if G is a bi-partite graph V_1, V_2, then we arrange the vertices that the adjacency matrix $A(G)$ has form $A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$, if X is an eigenvector corresponding to the eigenvalue λ, and the X' is obtained from X by changing the signs of the entries corresponding to vertices in V_2, then X' is an eigenvector corresponding to the value $-\lambda$, it follows that the spectrum of bi-partite graph is symmetric with respect to* 0.

Corollary 1.3 *The coefficients of the characteristic polynomial of a graph G satisfy:*

1. $c_1 = 0$;

2. $-c_2$ *is the number of edges of G;*

3. $-c_3$ *is twice the number of triangles in G;*

4. $c_4 = n_a - 2n_b$,n_a *is the number of pairs of disjoint edges in G,and* n_b *is the number of* $4-$*cycles in G.*

Proof. For each $i \in \{1, 2, \ldots, n\}$, the number $(-1)^i c_i$ is the sum of those principal minors of A which have i rows and columns. So we can argue as follows.

1. Since the diagonal elements of A are all zeros, $c_1 = 0$.
2. A principal minor with two rows and columns, and which has a non-zero entry, must be the form $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ there is one such minor for each pair of adjacent vertices of G, and each has value -1, Hence $(-1)^2 c_2 = -|E(G)|$, giving the result.
3. There are essentially three possibilities for non-trivial principal minors with three rows and columns: $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}$ and, of these, the only non-zero one is the last (whose value is 2). This principal minor corresponds to three mutually adjacent vertices in G, so we have the required description of c_3.
4. since the only elementary graph with four vertices are the C_4 and the graph only has two disjoint edges.

If two or more than two graphs has the same spectrum we call these graphs are co-spectrum, here give two graphs (Fig 1.11 and Fig 1.12) that has same spectrum.

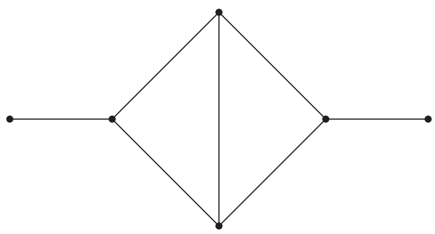

Fig 1.11 one of the two co-spectrum graphs

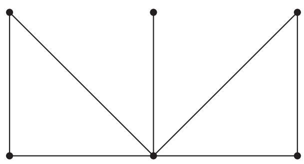

Fig 1.12 one of the two co-spectrum graph

Both of the above graphs has same characteristic polynomial: $\rho(G, \lambda) = \lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$

Theorem 1.14 (Rowlinson 1987) *Let G be a graph with a vertex v_1 of degree of 1, and let v_2 be the vertex adjacent to v_1, then $\rho(G, \lambda) = \lambda\rho(G - v_1) - \rho(G - v_1 - v_2)$.*

Corollary 1.4 *The characteristic polynomial of P_n is $\rho(P_n) = \lambda\rho(P_{n-1}) - \rho(P_{n-2})$.*

Theorem 1.15 (E.Hellbronner Spectra of graphs p.59) *Let G be the graph obtained by joining the vertex x of the graph G_1 to the vertex y of the graph G_2 by an edge. Let G_1', G_2' be the induced subgraph of G_1, G_2 obtained by deleting the vertex x, y from G_1, G_2, then*

$$\rho(G,\lambda) = \rho(G_1,\lambda)\rho(G_2,\lambda) - \rho(G_1',\lambda)\rho(G_2',\lambda)$$

In order to proof this theorem, we need more results from linear algebra, especially, the Laplacian expansion theorem of determinant. To extend the results of determinant and its applications in the graph theory, we may give the two important theorems of determinant they are the Laplacian theorem is given here and the Cauchy-Binet theorem is given in the next chapter.

Definition 1.9 *A minors M of order k is a sub-determinant obtained by selecting k rows and k columns of determinant D; and the $n-k$ minors left by delete a k-minors is called a co-minors, denoted as M'.*

Example: Let

$$D = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix}$$

If we select the first row and the third row, the first column and the third row, then we obtain a minor of order 2. That is

$$M' = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

and a co-minor of order 2 is

$$M' = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix}$$

Definition 1.10 *If $i_1, i_2, \ldots, i_k$ and $j_1, j_2, \ldots, j_k$ are the selected k rows and k columns, then $A = (-1)^{(i_1+\ldots+i_k)+(j_1+\ldots+j_k)}M$ is called a algebra co-minor of order k.*

In the above example

$$A = (-1)^{1+3+2+4}M' = 4$$

actually, Laplacian expansion theorem is more generalized determinant expand by a row or a column, now we give the Laplacian theorem.

Theorem 1.16 *$D = M_1A_1 + M_2A_2 + \ldots + M_sA_s$, where M and A are all minors and algebra co-minors of D with order k, $s = C(n,k) = \dfrac{n!}{k!(n-k)!}$. In other word M is all the minors of D with order k.*

In the above example, if we choose the first and the second row we obtain 6 minors,

$$M_{12}=\begin{vmatrix}1&2\\0&1\end{vmatrix};M_{13}=\begin{vmatrix}1&1\\0&2\end{vmatrix};M_{14}=\begin{vmatrix}1&4\\0&1\end{vmatrix};M_{23}=\begin{vmatrix}2&1\\1&2\end{vmatrix}$$

$$M_{24}=\begin{vmatrix}2&4\\1&1\end{vmatrix};M_{34}=\begin{vmatrix}1&4\\2&1\end{vmatrix}$$

associated algebra minors are:

$$A_{12}=(-1)^{1+2+1+2}\begin{vmatrix}0&0\\0&0\end{vmatrix}=0;A_{13}=(-1)^{1+2+1+3}\begin{vmatrix}0&1\\0&3\end{vmatrix}=0$$

$$A_{14}=(-1)^{1+2+1+4}\begin{vmatrix}0&1\\0&3\end{vmatrix}=0;A_{23}=(-1)^{1+2+2+3}\begin{vmatrix}0&2\\0&1\end{vmatrix}=0$$

$$A_{24}=(-1)^{1+2+2+4}\begin{vmatrix}1&1\\0&2\end{vmatrix}=0;A_{34}=(-1)^{1+2+3+4}\begin{vmatrix}1&2\\0&1\end{vmatrix}=0$$

then

$$D=M_{12}A_{12}+M_{13}A_{13}+M_{14}A_{14}+M_{23}A_{23}+M_{24}A_{24}+M_{34}A_{34}=-7$$

Proof. By applying the Laplacian development to the characteristic polynomial, we can easily get the result.

Also, readers may find the proof in book *Spectra of Graphs Theory and Application* on page 59.

Corollary 1.5 * *For $i=1,2,\ldots,n$ let G_i be the induced subgraph $G-v_i$, then*

$$\rho'(G,\lambda)=\sum_{i=1}^{n}\rho(G_i)$$

Proof. Row by row differentiation of $\rho(G,\lambda)=|\lambda I_n-A|$ yields the results.

Corollary 1.6 *If all sub-graph G_i are isomorphic with some graph H,then*

$$\rho'(G,\lambda)=n\rho(H,\lambda)$$

1.6 The spectrum of several graphs

The *complement* G of a graph G is the graph with the same vertex set, with two (distinct) vertices, adjacent in G if and only if these vertices are non-adjacent in G. The *direct sum* G_1+G_2 of graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, is a graph $G=(V,E)$ for which $V=V_1\cup V_2$,

$(V_1 \cup V_2{=}\emptyset)$, $E = E_1 \cup E_2$. The *complete product* $G_1 \bigtriangledown G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 + G_2$, by joining the every vertex of G_1 with the vertex of G_2. The *line graph* $L(G)$ of a graph G is constructed by taking the edges of G as vertices of $L(G)$ and joining two vertices in $L(G)$, whenever the corresponding edges in G have a common vertex.

Theorem 1.17([3]) *The incident and adjacent matrixes of G are M and A, and A_L is the adjacent matrix of $L(G)$, then*

1. $M^tM = A_L + 2I_m$*;*

2. if G is regular of degree k, then $MM^t = A + kI_n$*.*

In [5]*, Shu Jinlong developed this theorem to semi-regular bipartite graph. That is for* $V = V_1 + V_2$*, that* $d(v) = m|v \in V_1, d(v) = n|v \in V_2$*.*

Theorem 1.18 *$L(G)$ is k-regular if and only if G is regular or semi-regular bipartite graph.*

Lemma 1.2([7]) *Let* $A \in C^{m\times n}, B \in C^{n\times m}$*, then* $\lambda^n|(\lambda I_m - AB| = \lambda^m|\lambda I_n - BA|$.

Proof. For

$$\begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix}\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

caculate the determinant of this mutiplication of matrix, according to the propertices of determinant we get the result easily.

Lemma 1.3 *let* $U = \begin{pmatrix} \lambda I_n & -M \\ 0 & I_m \end{pmatrix}; V = \begin{pmatrix} I_n & M \\ M^t & \lambda I_m \end{pmatrix}$*, then*

$$\lambda^m \det(\lambda I_n - MM^t) = \lambda^n \det(\lambda I_m - M^tM)$$

Proof. Cause $\det(UV)$=$\det(VU)$, by calculating the above determinant we easily get the formula.

By the theorem 1.17 and lemma 1.2. We easily have the following theorem which is given by Sachs in 1976.

Theorem 1.19(Sachs 1967) *If G is a regular graph of degree k with n vertices and $m = nk/2$ edges, then*

$$\rho(L(G), \lambda) = (\lambda + 2)^{m-n}\rho(G, \lambda + 2 - k)$$

Proof. For $\rho(L(G), \lambda) = \det(\lambda I_m - A_L)$, by the theorem 1.17 and the lemma 1.3, we obtain this theorem easily.

We give an example by using this theorem, the line graph of K_n sometimes is called triangle graph, denoted as Δ_t. Since the spectrum of K_n is

$$SpecK_n = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

then

$$Spec\Delta_t = \begin{pmatrix} 2t-4 & t-4 & -2 \\ 1 & t-1 & t(t-3)/2 \end{pmatrix}$$

Theorem 1.20 *If λ is an eigenvalue of $L(G)$, then $\lambda \geqslant -2$.*

Proof. For any vector z, the inner product $\|Mz\|^2 = [Mz, Mz] = Mz^tMz = z^tM^tMz = z^t(A_L+2I_m)z$, since M^TM is a non negative symmetric matrix, then all eigenvalues are positive and the theorem is hold.

Theorem 1.21 *Let $G_1 \bigtriangledown G_2$ be the complete product of two simple graphs, then the complementary of this operation has following properties,*

$$\overline{G_1 \bigtriangledown G_2} = \overline{G_1} + \overline{G_2}$$

Theorem 1.22 *Let $G_1 + G_2$ be the union of G_1 and G_2, then*

$$\rho(G_1 + G_2, \lambda) = \rho(G_1, \lambda) \cdot \rho(G_2, \lambda)$$

Theorem 1.23 *Let $\rho(G_1 \bigtriangledown G_2)$ be the complete product of two simple graphs G_1 and G_2, then*

$$\rho(G_1 \bigtriangledown G_2, \lambda) = (-1)^{n_2}\rho(G_1, \lambda)\rho(\overline{G_2}, -\lambda-1) + (-1)^{n_1}\rho(G_2, \lambda)\rho(\overline{G_1}, -\lambda-1)$$

$$-(-1)^{n_1+n_2}\rho(\overline{G_1}, -\lambda-1)\rho(\overline{G_2}, -\lambda-1)$$

If G is a k-regular graph , then the polynomial $\rho(\overline{G}, \lambda)$ and $\rho(G_1 \bigtriangledown G_2, \lambda)$ are given by the following theorem.

Theorem 1.24(Sachs 1962) *If G is connected and regular of k, then* $\rho(\overline{G}, \lambda) = (-1)^n \dfrac{\lambda - n + k + 1}{\lambda + k + 1}\rho(G, -\lambda - 1)$.

This is a very useful tool to calculate the characteristic polynomial of certain type of graphs. We give an example below:

The graph obtained by deleting s disjoint edges from K_{2s} is called *cocktail party graph*, denoted as H_{2s}.

$$\rho(H_{2s}) = (-1)^{2s}\frac{\lambda + 2 - 2s}{\lambda + 2}\rho(sP_2, \lambda)$$

since the complement graph of a cocktail party graph is s disjoint edges K_2, and

$$\rho(P_2, \lambda) = (\lambda + 1)(\lambda - 1)$$

then

$$\rho(H_{2s}) = (-1)^{2s} \frac{\lambda + 2 - 2s}{\lambda + 2} [-\lambda(-\lambda - 2)]^s$$

Simplify this equation, We have

$$\rho(H_{2s}) = \frac{\lambda + 2 - 2s}{\lambda + 2} [(-\lambda)(-\lambda - 2)]^s$$

The spectrum of P_2 is

$$SpecP_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

then the spectrum of H_{2s} is

$$SpecH_{2s} = \begin{pmatrix} 2s-2 & 0 & -2 \\ 1 & s & s-1 \end{pmatrix}$$

If G_1 is a r_1 regular graph and G_2 is a r_2 regular graph, then the characteristic polynomial of the complete product of these two graphs is given by the following theorem.

Theorem 1.25 *The characteristic polynomial of complete product of two graphs is,* $\rho(G_1 \triangledown G_2, \lambda) = \dfrac{\rho(G_1, \lambda)\rho(G_2, \lambda)}{(\lambda - r_1)(\lambda - r_2)} [(\lambda - r_1)(\lambda - r_2) - n_1 n_2]$.

Corollary 1.7 *Suppose that a graph has two vertices v_i and v_j has same neighbor vertices Γ, then the vector X whose only non-zero entries are $x_i = 1$ and $x_j = -1$ is an eigenvector of the adjacency matrix with eigenvalue 0, if Γ has r vertices, then the multiplicity is at least $r - 1$.*

Theorem 1.26 *If the spectrum of the graph G contains an eigenvalue λ_0 with multiplicity $p > 1$, then the spectrum of the complementary graph $\overline{G}$ contains an eigenvalue $-\lambda_0 - 1$ with multiplicity q,where $p - 1 \leqslant q \leqslant p + 1$.*

1.7 Results from matrix theory

In this section, We study several characteristic polynomials of special graphs. We also study the Rayleigh quotient of vectors and the largest minimal eigenvalue of matrix. In the end of this section, we study the properties of circular matrix.

1. Empty graph G with n vertices its characteristic polynomial, $\rho(G, \lambda) = \lambda^n$.
2. $\rho(K_n, \lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}$.

3. each component of a regular graph of degree 1 is isomorphic with graph K_2,then k copy of K_2 is $\rho(K_n, \lambda) = (\lambda^2 - 1)^k$.
4. $\rho(K_{n_1,n_2}, \lambda) = (\lambda^2 - n_1 n_2)\lambda^{n_1+n_2-2}$.
5. $\rho(K_{1,n}, \lambda) = (\lambda^2 - n)\lambda^{n-1}$.
6. $\rho(P_n, \lambda) = \sum_{k=0}^{n/2}(-1)^k \begin{pmatrix} n-k \\ k \end{pmatrix} \lambda^{n-2k} = U_n(\lambda/2)$. Where

$$U_n(x) = \frac{\sin[(n+1)\arccos(x)]}{\sqrt{1-x^2}}$$

is the Chebyshev polynomial of the second.

7. $\rho(C_n, \lambda) = -2 + \sum_{k=0}^{[n/2]}(-1)^k \frac{n}{n-k} \begin{pmatrix} n-k \\ k \end{pmatrix} \lambda^{n-2k} = 2\cos(n \arccos x/2) - 2$
8. $\rho(K_{\frac{n}{k},\dots,\frac{n}{k}}, \lambda) = \lambda^{n-k}(\lambda + \frac{n}{k} - n)(\lambda + \frac{n}{k})^{k-1}$

In order to take a review of calculating the determinant, we give proof for 2 and 5.

Proof. Since the characteristic polynomial of K_n is:

$$\rho(K_n, \lambda) = \begin{vmatrix} \lambda & -1 & \dots & -1 \\ -1 & \lambda & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & \lambda \end{vmatrix}$$

by the properties of determinant, we add all the other rows to the first row and take the common factor $\lambda - (n-1)$ then

$$\rho(K_n, \lambda) = (\lambda - n + 1)\begin{vmatrix} 1 & 1 & \dots & 1 \\ -1 & \lambda & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & \lambda \end{vmatrix}$$

add the first row to each other rows, finally, we have formula 2.

Proof. Since the characteristic polynomial of $K_{1,n}$ is:

$$\rho(K_{1,n}, \lambda) = \begin{vmatrix} \lambda & -1 & -1 & \dots & -1 \\ -1 & \lambda & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & \lambda \end{vmatrix}$$

multiple the first row by $1/\lambda$ then add to every other rows, then expand this new determinant by the first row then we get a symmetric determinant of the diagonal entries are $\lambda - 1/\lambda$ and all other entries are $-1/\lambda$. With the skill to calculate the symmetric determinant, we can easily get the formula 5.

Some spectrum of certain graphs:

1. $SpecK_n = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$;

2. $Spec(K_{a,b}) = \begin{pmatrix} \sqrt{ab} & 0 & -\sqrt{ab} \\ 1 & a+b-2 & 1 \end{pmatrix}$;

3. $SpecP_n = \begin{pmatrix} 2\cos\pi/(n+1) & 2\cos 2\pi/(n+1) & \dots & 2\cos n\pi/(n+1) \\ 1 & 1 & \dots & 1 \end{pmatrix}$;

4. $SpecC_n = \begin{pmatrix} 2 & 2\cos 2\pi/n & \dots & 2\cos(n-1)\pi/n \\ 1 & 2 & \dots & 2 \end{pmatrix}$ $(n\ is\ odd)$;

5. $SpecC_n = \begin{pmatrix} 2 & 2\cos 2\pi/n & \dots & 2\cos(n-2)\pi/n & -2 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix}$ $(n\ is\ even)$;

especially,

$$SpecP_4 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$SpecC_{10} = \begin{pmatrix} 2 & \frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} & -2 \\ 1 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

Sometimes we call $K_{1,n}$ is a star. According to the expression of complete bipartite graph's characteristic polynomial, we can easily have:

$$Spec(K_{1,n}) = \begin{pmatrix} \sqrt{n} & 0 & -\sqrt{n} \\ 1 & n-1 & 1 \end{pmatrix}$$

Spectrum of peterson graph is $SpecO_3 = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$

We cite two important result from linear algebra. the proposition 1.1 is the relation between the coefficients of the characteristic polynomial of a matrix and its principle minors; the second result is about the circular matrix.

Proposition 1.1 *The coefficient of the characteristic a_i equals the summation over all principal minors of order i multiple by $(-1)^{n-i}$.*

An $n \times n$ matrix is said to be cyclic matrix if its $i-th$ row is obtained by a cyclic shift $i-1$ steps of the first row, that is the cyclic matrix is determined by the first row.

$$w = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}; \quad w^2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$s = \begin{pmatrix} s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_n \\ s_n & s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_3 & s_4 & s_5 & \cdots & s_1 & s_2 \\ s_2 & s_3 & s_4 & \cdots & s_n & s_1 \end{pmatrix}; \quad s = \sum_{j=1}^{n} s_j w^{j-1}$$

Since the eigenvalues of w is $1, \omega, \omega^2, \omega^{n-1}$, where $\omega = \exp(2\pi i/n)$ it follows that the eigenvalues of s are $\lambda_r = \sum_{j=1}^{n} s_j \omega^{(j-1)r}, r = 0, 1, \ldots, n-1$.

Since the adjacent matrix of a cycle C_n is a circular matrix generated by the first row $r_1 = [0, 1, 0, \cdots, 1]^{\mathrm{T}}$, then we can easily obtain the spectrum of C_n by the above properties of circular matrix.

1.8 About the largest zero of characteristic polynomials

Lemma 1.4 *If the distinct eigenvalues of $A(G)$ are $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ then $\lambda_1 \leqslant \sqrt{\dfrac{2\varepsilon(n-1)}{n}}$.*

Proof. By the lemma 1.1 and the summation of the eigenvalues of a matrix is $-a_1$ the coefficient of the characteristic polynomial. We have:

$$\sum_{i=1}^{n} \lambda_i = -c_1 = 0$$

$$\sum_{i \neq j} \lambda_i \lambda_j = -\varepsilon$$

$$\sum_{i=1}^{n}\lambda_i^2 = (\sum_{i=1}^{n}\lambda_i)^2 - 2\sum_{i\neq j}\lambda_i\lambda_j = -2\varepsilon$$

$$\lambda_1 = -(\lambda_2 + \lambda_3 + \ldots + \lambda_n)$$

$$\lambda_1^2 = 2\varepsilon - (\lambda_2^2 + \ldots + \lambda_n^2)$$

By *Cauchy-Schwaze* inequality we have:

$$|\lambda_2 + \lambda_3 + \ldots + \lambda_n| \leqslant \sqrt{\lambda_2^2 + \lambda_3^2 + \ldots + \lambda_n^2}\sqrt{n-1}$$

$$\lambda_1^2 \leqslant (2\varepsilon - \lambda_1^2)(n-1)$$

$$\lambda_1 \leqslant \sqrt{(2\varepsilon - \lambda_1^2)(n-1)}$$

In fact, this theorem also can be proved by the corollary 2.2 and the matrix theorem. Another bound of the same type is $\lambda_1 \leqslant \sqrt{2\varepsilon - \nu + 1}$ the equation hold if and only if a component is a complete graph or a star and other components are k_2 that are isolated edges. (East China Normal University Hongyuan 1988). If graph G have m edges and l isolated vertices, then $\lambda_1 \leqslant \sqrt{2\varepsilon - \nu + l + 1}$. (SHU Jin-long in 2000). Vladimir Nikiforov, Eigenvalues and degree deviation in graphs, linear Algebra and Application $414(2006)347 - 360$ gives another bound of λ_1, let $s(G) = \sum\limits_{v\in V(G)} |d(v) - 2m/n|$, then

1. $s^2/2n^2\sqrt{2m} \leqslant \lambda_1 - 2m/n \leqslant \sqrt{s(G)}$;
2. $\lambda_k(G) + \lambda_{n-k+2}(G^c) \geqslant -1 + 2\sqrt{2s}$, for all $2 \leqslant k \leqslant n$;
3. $\lambda_n + \lambda_n(G^c) \leqslant -1 - s^2/2n^3$.

Theorem 1.27 *Let G be a regular graph of degree k, then*

1. k is an eigenvalue of G;

2. If G is connected, then the multiplicity of k is 1;

3. For any eigenvalue λ of G, We have $\mid \lambda \mid \leqslant k$.

Proof. 1. If $u = (1, 1, \ldots, 1)^t$, A is the adjacent matrix of G, then we have $Au = ku$, since there are k 1's in each row, thus k is an eigenvalue of G.

2. Let $x = (x_1, x_2, \ldots, x_n^t)$ denotes an non-zero vector for which $Ax = kx$ and suppose that x_j is an entry of x with the largest absolute value. Since $(Ax)_j = kx_j$, we have $\Sigma x_i = kx_j$, where Σ denotes summation over those k vertices v_i which are adjacent to all those vertices are adjacent to v_j, by the maximal property of x_j. It follows that $x_i = x_j$, for all those vertices if G is connected we may proceed successively in this way, eventually show that all entries of x are equal, thus x is a multiple of u and the space of eigenvalue associated with the eigenvalue k has dimension 1.

3. Suppose that $Ay = \lambda y, y \neq 0$ and let y_j denote an entry of y which is largest in absolute value. By the same argument as in (2), we have $\Sigma y_i = \lambda y_j$ and so

$$| \lambda || y_j |=| \Sigma y_i | \leqslant \Sigma | y_j | \leqslant k | y_j |$$

thus $| \lambda | \leqslant k$, as required.

We need a useful technique from matrix theory.

Theorem 1.28 *(Schur) For any matrix A of order n there exist a U matrix of order n and an upper triangle matrix R, that*

$$U^H AU = R \quad or \quad A = URU^H$$

hold. Where the diagonal entries are the eigenvalues of A.

Corollary 1.8 *If A is a Hermit matrix, then $A \sim \Lambda$, the diagonal entries are the eigenvalues of matrix A.*

Let (x, y) denotes the inner product of the column vectors x, y. For any real $n \times n$ symmetric matrix X and any real non-zero $n \times 1$ column vector z, the number $(z, Xz)/(z, z)$ is know as the *Rayleigh quotient*, and written $R(X, z)$ in matrix theory it is proved that

$$\lambda_{\max}(X) \geqslant R(X, z) \geqslant \lambda_{\min}(X) \quad for \quad all \quad z \neq 0$$

a result which has important applications in graph theory. Here, we give a simple proof.

Proof. since X is a real symmetric matrix, then there is a matrix $U^{\mathrm{T}} = U^{-1}$ that $U^{\mathrm{T}}XU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ holds, suppose that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$, now

$$\lambda_1 I - X = U^{\mathrm{T}}(\lambda_1 I - \mathrm{diag}(\lambda_1, \ldots, \lambda_n))U \geqslant 0$$

cause $\lambda_1 = \max\{\lambda_1, \ldots, \lambda_n\}$ then $\lambda_1 - \lambda_i \geqslant 0, i \in (1, \ldots, n)$. $\lambda_1 I \geqslant X$. The other inequality prove in similar way.

Theorem 1.29 *Suppose that A is a Hermit matrix of order n, eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$, then*

1. $R(kx) = R(x), k \in C, k \neq 0$;

2. $\lambda_n \leqslant R(x) \leqslant \lambda_1$;

3. $\lambda_1 = \max R(x), \lambda_n = \min R(x), x \neq 0$.

Theorem 1.30 *Let $x_1, x_2, \ldots, x_n$ be the eigenvectors correspond to eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, and $V_i^j = Span\{x_i, x_{i+1}, \ldots, x_j\}$, then*

$$\lambda_i = \max R(x), \quad \lambda_j = \min R(x), \quad x \in V_i^j, \quad x \neq 0$$

Proof. $\forall x \in V_i^j$ and $x \neq 0$, that $x = a_i x_i + \ldots + a_j x_j$, then

$$\lambda_j \leqslant R(X) = \frac{|a_i|^2\lambda_i + |a_{i+1}|^2\lambda_{i+1} + \ldots + |\lambda_j|^2\lambda_j}{|a_i|^2 + |a_{i+1}|^2 + \ldots + |a_j|^2} \leqslant \lambda_i$$

especially. If $x = x_i$, then $R(x) = \lambda_i$, if $x = x_j$, then $R(x) = \lambda_j$. So the theorem holds.

Theorem 1.31 *Suppose that A and E are the Hermit matrices of order n, $B = A + E$, and the eigenvalues of A, B and E are $\lambda_1 \geqslant \ldots \geqslant \lambda_n$, $\mu_1 \geqslant \ldots \geqslant \mu_n$ and $\varepsilon_1 \geqslant \ldots \geqslant \varepsilon_n$, then*

$$\lambda_i + \varepsilon_n \leqslant \mu_i \leqslant \lambda_i + \varepsilon_1, i = 1, 2, \ldots, n$$

Proposition 1.2 *1. If H is an induced subgraph of G, then*

$$\lambda_{\max}(H) \leqslant \lambda_{\max}(G); \lambda_{\min}(H) \geqslant \lambda_{\min}(G)$$

2. If the greatest and least degrees among the vertices of G are $d_{\max}(G), d_{\min}(G)$, and the average degree is $d_{\text{ave}}(G)$, then

$$d_{\max}(G) \geqslant \lambda_{\max}(G) \geqslant d_{\text{ave}}(G) \geqslant d_{\min}(G)$$

Proof. 1. We may suppose that the vertices of G are labeled so that the adjacency matrix of G has a leading principal minors A_0, which is the adjacency matrix of subgraph H. Let z be chosen such that $A_0 z_0 = \lambda_{\max} z_0$ and $(z_0, z_0) = 1$. Furthermore, let z be column vector with $|V(H)|$ rows formed by adding zero entries to vector z_0. Then

$$\lambda_{\max}(A_0) = R(A_0, z_0) = R(A, z) \leqslant \lambda(A)$$

that is,

$$\lambda_{\max}(H) \leqslant \lambda_{\max}(G)$$

The other inequality is proved similarly.

2. Let u be the column vector each of whose entries are $+1$, then if $n = |V(G)|$ and d_i is the degree of the vertex v_i, we have

$$R(A, u) = (\sum_{i,j} a_{ij})/n = \sum_i d_i = d_{\text{ave}}$$

the Rayleigh quotient $R(A, u)$ is at most $\lambda_{\max}(A)$ that is $\lambda_{\max}(G)$, and it is clear that the average degree is not less than the minimum degree. Hence

$$\lambda_{\max}(G) \geqslant d_{\text{ave}} \geqslant d_{\min}(G)$$

Finally, let X be an eigenvector corresponding to the eigenvalue $\lambda_0 = \lambda_{\max}(G)$, and let x_j be a largest positive entry of X. By an argument similar to the used in (1) we have

$$\lambda_0 x_j = (\lambda_0 X)_j = (AX)_j = \sum x_i \leqslant d_j x_j \leqslant d_{\max}(G) x_j$$

where the sum $\sum$ is take over the vertices v_i adjacent to v_j, Thus $\lambda_0 \leqslant d_{\max}(G)$.

Proposition 1.3([3]) *Let X be a symmetric matrix, partitioned in the form $X = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$, where P and R are square symmetric matrix, then*

$$\lambda_{\max}(X) + \lambda_{\min}(X) \leqslant \lambda_{\max}(P) + \lambda_{\max}(R)$$

Proof. Let $\lambda = \lambda_{\min}(X)$ and take an arbitrary $\varepsilon \geqslant 0$, then $X' = X - (\lambda - \varepsilon)I$ is a positive definite symmetric matrix, partitioned in the same way as X, with

$$P' = P - (\lambda - \varepsilon)I, \quad Q' = Q, \quad R' = R - (\lambda - \varepsilon)I$$

By appling the method of Rayleigh quotient to matrix X', it can be show that

$$\lambda_{\max}(X') \leqslant \lambda_{\max}(P') + \lambda_{\max}(R')$$

Thus, in terms of X, P and R, we have

$$\lambda_{\max}(X) - (\lambda - \varepsilon) \leqslant \lambda_{\max}(P) - (\lambda - \varepsilon) + \lambda_{\max}(R) - (\lambda - \varepsilon)$$

and since the arbituary of ε and $\lambda = \lambda_{\max}(X)$, we have the result.

Theorem 1.32 *Let A be a real symmetric matrix, partitioned into t^2sub-matrix A_{ij} in such that a way that the row and the column partitioned in the same, in other words, each diagonal sub-matrix A_{ii} are square, then*

$$\lambda_{\max}(A) + (t-1)\lambda_{\min}(A) \leqslant \sum_{i=1}^{t} \lambda_{\max}(A_{ii})$$

Proof. We proof this result by induction on t. It is true when $t = 2$, by the proposition 1.3 Suppose that it is true when $t = T - 1$, then we show that is hold when $t = T$. Let A be partitioned into t^2 sub-matrixes,in the manner stated, B be the matrix with the last row and the last column deleted. By the proposition1.30,

$$\lambda_{\max}(A) + \lambda_{\min}(A) \leqslant \lambda_{\max}(B) + \lambda_{\max}(A_{TT})$$

and by the induction hypothesis,

$$\lambda_{\max}(B) + (T-2)\lambda_{\min}(B) \leqslant \sum_{i=1}^{T-1} \lambda_{\max}(A_{ii})$$

now $\lambda_{\min}(B) \geqslant \lambda_{\min}(A)$, thus adding the two inequalities,we have the result for $t = T$, and the general result follows by induction.

Let $v_1, v_2, \ldots, v_n$ be orthogonal unit eigenvectors to the eigenvalues of Laplacian matrix Q of G, then

$$u_k = \min_{\|x\|=1, x \perp Span\{v_1,\ldots,v_{k-1}\}} \langle Qx, x\rangle;$$

$$\lambda_k = \min_{M \subset \Re^n, dim(M)=k-1} \{\max_{\|x\|=1, x\perp(M)} \langle Ax, x\rangle\}$$

this result we can find in *R.Horn, C.Johnson, Matrix Analysis, Cambridge University Press, Cambridge,* 1985*, p:561.*

Theorem 1.33 *Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be the eigenvalues of adjacency matrix A, and $0 = u_1 \leqslant u_2 \leqslant \ldots \leqslant u_n$, then $\delta(G) \leqslant \lambda_k + u_k \leqslant \varDelta$, and $\lambda_k(G) + \lambda_{n-k+2}(G^c) \geqslant \delta(G) - \varDelta(G) - 1$, for $2 \leqslant k \leqslant n$.*

Proof. Let y be a vector associate with the largest eigenvalue of A and $\|y\| = 1$ and also $y \perp Span\{v_1, \ldots, v_{k-1}\}$. Let $y = (y_1, \ldots, y_n)$, we find that

$$u_k = \langle Qy, y\rangle = \sum_{v\in V(G)} d(v)y_v^2 - \langle Ay, y\rangle$$

$$\leqslant \varDelta(G) - \max_{\|x\|=1, x\perp Span\{v_1,\ldots,v_{k-1}\}} \langle Ax, x\rangle$$

$$\leqslant \min_{M\subset\Re^n, dim(M)=k-1} \{\max_{\|x\|=1, x\perp(M)} \langle Ax, x\rangle\}$$

$$= \varDelta(G) - \lambda_k(G)$$

As to the second inequality, it is well know that $u_k(G) + u_{n-k+2}(G^c) = n$ for all $2 \leqslant k \leqslant n$,

$$n + \lambda_k + \lambda_{n-k+2}(G^c)$$

subsitude $u_k(G) + u_{n-k+2}(G^c) = n$ into above equation we have

$$\geqslant \delta(G) + \delta(G^c) \geqslant \delta(G) + n - 1 - \varDelta(G)$$

simplify it we will get the result.

Theorem 1.34 *Let $\lambda_2(G)$ be the second largest zero of G and $\lambda_n(G^c)$ be the smallest zero of G^c, then $\lambda_2 + \lambda_n(G^c) \leqslant -1$.*

Readers can find Vladimir Nikiforov, Eigenvalues and extremal degrees of graphs,

Linear Algebra and its Applications 419(2006)735 − 738.

Theorem 1.35 *[Courant-weyl inequlity] Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be the eigenvalues of a symmetric real matrix, if A, B and C are all the symmetric real matrixes and $C = A + B$, then $\lambda_{n-j-i}(C) \geqslant \lambda_{n-i}(A) + \lambda_{n-j}(B)$.*

1.9 Spectrum radius

Suppose that $A = (a_{ij}) \in R^{m\times n}$, if $a_{ij} \geqslant 0, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$, then we call A is a non-negative matrix, write as $A \geqslant 0$. If $a_{ij} > 0$, we call A is positive matrix, write as $A > 0$. A transpose matrix $P = \prod E(i, j)$, that is a serious elementary operation of the first type.

Definition 1.11 *A matrix is reducible if there is a transpose matrix such that*

$$PAP^{\mathrm{T}} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where A_{11} is a square matrix of order r, A_{22} is a square matrix of order $n - r$. Otherwise, A is irreducible.

Non-negative matrix has two types, one is reducible matrix and another is irreducible matrix. In 1907, *Perron* proved an important theorem about the relation between the positive matrix's largest eigenvalue and the correspondence eigenvector. We call the largest eigenvalue is *spectrum radius (λ_1)* and the eigenvector correspond to largest value is *Perron vector*, then we have several theorems below:

Theorem 1.36(Perron) *Suppose that A is a positive matrix, then*

1. $\lambda_1 > 0$, and the Perron vector $x > 0, x \in R^n$;

2. all other eigenvalue $|\lambda_i| < \lambda_1$;

3. λ_1 has multiplicity one.

Theorem 1.37(Perron-Frobenius) *If a connected graph G with at least two vertices,then*

1. $\lambda_1 > 0$,with multiplicity l;

2. there exists a unique positive unit eigenvector corresponding to λ_1;

3. all other eigenvalue $|\lambda_i| < \lambda_1$;

4. deleting any edge from G, λ_1 will decrease.

Theorem 1.38([20]) *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G, and the $u_1, u_2, \ldots, u_{n-1}$ be eigenvalues of $G - u$,then*

$$\lambda_1 \geqslant u_1 \geqslant \lambda_2 \geqslant u_2 \geqslant \ldots \geqslant u_{n-1} \geqslant \lambda_n$$

Theorem 1.39([21]) *Let G be a connected graph and λ_1 be the spectral radius of $A(G)$, u and v be two vertices of G. Suppose that $\{v_1, v_2, \ldots, v_s\} \in N(v)\backslash N(u), (1 \leqslant s \leqslant d(v))$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$. Let G^* be the graph obtained by deleting the edges (vv_i) and adding the edges $uv_i, (1 \leqslant i \leqslant s)$, then $\lambda_1(G) < \lambda_1(G^*)$.*

Theorem 1.39 indicates that the spectral radius is increase when the graph becomes more concentrate, the star has the largest eigenvalue, that is $\sqrt{n-1}$ and the path has the smallest eigenvalue, that is $2\cos\dfrac{\pi}{n+1}$, and for matching polynomial hold as well. Reads may refer to *Bolian Liu,* 2005*, Combinatorial Matrix Theory.*

Exercise

In this book, if we not specially inform you, take all graphs as simple connected graphs.

1. If G is a connected simple graph of order n, then $\varepsilon \leqslant \begin{pmatrix} n \\ 2 \end{pmatrix}$.
2. under the condition of exercise 1's condition $\varepsilon \leqslant \begin{pmatrix} n \\ 2 \end{pmatrix}$ if and only if $G \cong K_n$.
3. If $\varepsilon(K(m,n)) = mn$.
4. If G is a connected simple graph of order n, then $\varepsilon \leqslant n^2/4$.
5. We say a graph is self-complement, write as $G \cong G^c$. If G is self-complement then $n \equiv 0, 1(\text{mod}4)$.
6. ♣ If G is a simple graph and the eigenvalues of $A(G)$ are different, then the automorphism group of G is commutative.
7. Every simple graph of order n is isomorphism to a sub graph of K_n.
8. Every subgraph of bipartite graph is bipartite graph.
9. ♣ G is a simple graph, for any integer number n for $1 < n < \nu - 1$, if $\nu \geqslant 4$ and all induced subgraph of order n has same numbers of edges, then $G \cong K_\nu$ or $G \cong K_\nu^C$.
10. Prove that $\delta \leqslant 2\varepsilon/\nu \leqslant \Delta$.
11. If G is k-regular bipartite graph ($k > 0$) have an vertex departing (X, Y), then $|X| = |Y|$.
12. Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be the eigenvalues of $A(G)$, then $\lambda_1 \leqslant \sqrt{2\varepsilon(n-1)/n}$.
13. For a simple graph, positive integer sequence $(8, 6, 5, 4, 3, 2, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not a graph's degree sequence.
14. For a simple graph, if $A(G)$ has n different eigenvalues, then the automorphisms group $\Gamma(G)$ is Abelian.

Group Project

For any simple graph G and a positive integral m, $(m \leqslant n)$, n is the order of the graph, there exist a m partite spanning subgraph H, which satisfy the inequality

$$(1 - \frac{1}{m})\varepsilon(G) \leqslant \varepsilon(H) \leqslant T_{m,n}$$

where $T_{m,n}$ is the Turan graph, how to find this spanning bipartite graph?